Escape rate of the Brownian motions on hyperbolic spaces

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Abstract: We discuss the escape rate of the Brownian motion on a hyperbolic space. We point out that the escape rate is determined by using the Brownian expression of the radial part and a generalized Kolmogorov's test for the one dimensional Brownian motion.

Key words: Escape rate; Brownian motion; hyperbolic space.

1. Introduction. Let $\mathbf{H}^d$ be the d-dimensional hyperbolic space and $\mathbf{M} = (\{X_t\}_{t\geq 0}, \{P_x\}_{x\in\mathbf{H}^d})$ the Brownian motion on $\mathbf{H}^d$ generated by the half of the Laplace-Beltrami operator. For a fixed point $o \in \mathbf{H}^d$, define $P = P_o$ and $R_t = d(o, X_t)$, where d is the distance function of $\mathbf{H}^d$. In this note, we show

Theorem 1.1. *Let $g(t)$ be a positive function on $(0,\infty)$ such that for some $t_0 > 0$, $\sqrt{t}g(t)$ is nondecreasing and $g(t)/\sqrt{t}$ is bounded for all $t \geq t_0$.*

(i) *For the function $r_1(t) := (d-1)t/2 + \sqrt{t}g(t)$,*

$$P(\text{there exists } T > 0 \text{ such that } R_t < r_1(t) \text{ for all } t \geq T) = 1 \text{ or } 0 \tag{1.1}$$

according as

$$\int^{\infty} (1 \vee g(t)) e^{-g(t)^2/2} \frac{\mathrm{d}t}{t} < \infty \text{ or } = \infty. \tag{1.2}$$

(ii) *For the function $r_2(t) := (d-1)t/2 - \sqrt{t}g(t)$,*

$$P(\text{there exists } T > 0 \text{ such that } R_t > r_2(t) \text{ for all } t \geq T) = 1 \text{ or } 0 \tag{1.3}$$

according as (1.2) *holds.*

The function $r_1(t)$ is called an *upper rate function* for $\mathbf{M}$ if the probability in (1.1) is 1. By the same way, the function $r_2(t)$ is called a *lower rate function* for $\mathbf{M}$ if the probability in (1.3) is 1. According to Theorem 1.1, we have for $c > 0$,

- the function $r(t) := (d-1)t/2 + \sqrt{ct\log\log t}$ is an upper rate function for $\mathbf{M}$ if and only if $c > 2$;
- the function $r(t) := (d-1)t/2 - \sqrt{ct\log\log t}$ is a lower rate function for $\mathbf{M}$ if and only if $c > 2$.

For the Brownian motions on Riemannian manifolds, more generally symmetric diffusion processes generated by regular Dirichlet forms, upper and lower rate functions are given in terms of volume growth rate ([1–4,6,11]). As for the upper rate functions, the results in [2–4,6,11] are applicable to the Brownian motions on Riemannian manifolds with exponential volume growth rate, as to $\mathbf{M}$; however, as for the lower rate functions, the results in [1–3] are not applicable to $\mathbf{M}$ because the doubling condition is imposed on the volume growth. Grigor'yan and Hsu [4] also discussed the sharpness of the upper rate functions for $\mathbf{M}$ or for the Brownian motion on a model manifold, that is, a spherically symmetric Riemannian manifold with a pole. Using the fact that

$$\lim_{t\to\infty} \frac{R_t}{t} = \frac{d-1}{2}, \quad P\text{-a.s.} \tag{1.4}$$

(which follows from (2.2) below), they remarked that the function $r(t) = ct$ is an upper rate function for $\mathbf{M}$ if $c > (d-1)/2$, and not if $0 < c < (d-1)/2$. This observation is still valid for the lower rate functions. See also [7] for the result of the law of the iterated logarithms-type to the Brownian motions on model manifolds.

For the proof of Theorem 1.1, we make use of the Brownian expression of the radial part R_t ((2.2) below) as in [4,7], together with a generalized version of Kolmogorov's test for the one dimensional Brownian motion ([9,10]). In fact, the integral in (1.2) is the same with that in this test. The assumption on $g(t)/\sqrt{t}$ will be needed in (2.7) and (2.8) below.

2. Proof of Theorem 1.1. Let $\mathbf{B} = (\{B_t\}_{t\geq 0}, P)$ be the one dimensional Brownian motion starting from the origin. Then a generalized

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Kolmogorov's test holds:

Theorem 2.1 ([9, Theorem 3.1 and Lemma 3.3] and [10, Theorem 2.1]). *Under the full conditions of Theorem* 1.1,

$$P(\textit{there exists } T > 0 \textit{ such that } |B_t| < \sqrt{t}g(t) \textit{ for all } t \geq T) = 1 \textit{ or } 0 \tag{2.1}$$

according as (1.2) *holds. This assertion is valid even if* $|B_t|$ *in the equality above is replaced by* B_t *or* $-B_t$.

By comparison with Kolmogorov's test (see, e.g., [8, 4.12]), we do not need to assume that $g(t) \nearrow \infty$ as $t \to \infty$ in Theorem 2.1.

Proof of Theorem 1.1. Recall that $\mathbf{M} = (\{X_t\}_{t\geq 0}, P)$ is the Brownian motion on $\mathbf{H}^d$ starting from a fixed point $o \in \mathbf{H}^d$ and $R_t = d(o, X_t)$ is the radial part of X_t. Then by [5, Example 3.3.3],

$$R_t = B_t + \frac{d-1}{2}\int_0^t \coth R_s \, \mathrm{d}s. \tag{2.2}$$

Assume the full conditions of Theorem 1.1. We first discuss the lower bound of R_t. Since $\coth x \geq 1$ for any $x > 0$, we obtain by (2.2),

$$R_t \geq B_t + \frac{d-1}{2}t \quad \text{for any } t \geq 0. \tag{2.3}$$

Hence if the integral in (1.2) is convergent, then the probability in (1.3) is 1 by Kolmogorov's test. By the same way, if the integral in (1.2) is divergent, then the probability in (1.1) is 0. □

We next discuss the upper bound of R_t. Since $B_t = o(t)$ as $t \to \infty$, we see by (2.3) that there exists $c > 0$ such that $P(A) = 1$ for

$$A := \{\text{there exists } T_1 > 0 \text{ such that } R_t \geq ct \text{ for all } t \geq T_1\}. \tag{2.4}$$

Under the event A,

$$\coth R_s - 1 = \frac{2}{e^{2R_s} - 1} \leq \frac{2}{e^{2cs} - 1}$$

for any $s \geq T_1$, which implies that for all $t \geq T_1$,

$$\begin{aligned}\int_0^t (\coth R_s - 1)\, \mathrm{d}s &= \int_0^{T_1} (\coth R_s - 1)\, \mathrm{d}s + \int_{T_1}^t (\coth R_s - 1)\, \mathrm{d}s \\ &\leq \int_0^{T_1} (\coth R_s - 1)\, \mathrm{d}s + \int_{T_1}^\infty \frac{2}{e^{2cs} - 1}\, \mathrm{d}s =: C_{T_1}.\end{aligned}$$

Since there exists an integer valued random variable N such that

$$\frac{d-1}{2} C_{T_1} \leq N, \tag{2.5}$$

we obtain for such N,

$$\begin{aligned}R_t &= B_t + \frac{d-1}{2}t + \frac{d-1}{2}\int_0^t (\coth R_s - 1)\, \mathrm{d}s \\ &\leq B_t + \frac{d-1}{2}t + \frac{d-1}{2}C_{T_1} \\ &\leq B_t + \frac{d-1}{2}t + N\end{aligned} \tag{2.6}$$

for all $t \geq T_1$.

Assume first that the integral in (1.2) is convergent. Then there exists a positive constant c_n for each $n \geq 1$ such that the function $h_1^{(n)}(t) := g(t) - n/\sqrt{t}$ satisfies

$$\begin{aligned}\int_\cdot^\infty (1 \vee h_1^{(n)}(t)) e^{-h_1^{(n)}(t)^2/2} \frac{\mathrm{d}t}{t} \\ \leq c_n \int_\cdot^\infty (1 \vee g(t)) e^{-g(t)^2/2} \frac{\mathrm{d}t}{t} < \infty.\end{aligned} \tag{2.7}$$

Hence Theorem 2.1 implies that for each $n \geq 1$,

$$P(\text{there exists } T > 0 \text{ such that } |B_t| < r_1^{(n)}(t) \text{ for all } t \geq T) = 1$$

for $r_1^{(n)}(t) := \sqrt{t}h_1^{(n)}(t) (= \sqrt{t}g(t) - n)$. In particular, we get $P(B_1) = 1$ for

$$B_1 := \{\text{for each } n \geq 1, \text{ there exists } S_n > 0 \text{ such that } |B_t| < r_1^{(n)}(t) \text{ for all } t \geq S_n\}.$$

Under the event $A \cap B_1$, since there exists $T_2 > 0$ for $N \geq 1$ in (2.5) such that

$$B_t < r_1^{(N)}(t) = \sqrt{t}g(t) - N \quad \text{for all } t \geq T_2,$$

we have by (2.6),

$$R_t < \frac{d-1}{2}t + \sqrt{t}g(t) \quad \text{for all } t \geq T_1 \vee T_2.$$

Therefore, the probability in (1.1) is 1.

Assume next that the integral in (1.2) is divergent. Then by the same way as in (2.7), the function $h_2^{(n)}(t) := g(t) + n/\sqrt{t}$ satisfies for each $n \geq 1$,

$$\int_\cdot^\infty (1 \vee h_2^{(n)}(t)) e^{-h_2^{(n)}(t)^2/2} \frac{\mathrm{d}t}{t} = \infty. \tag{2.8}$$

Hence Theorem 2.1 yields that for each $n \geq 1$,

$$P(\text{for any } t > 0, \text{ there exists } T \geq t \text{ such that } B_T \leq -r_2^{(n)}(T)) = 1$$

for $r_2^{(n)}(t) := \sqrt{t}h_2^{(n)}(t)(= \sqrt{t}g(t) + n)$. In particular, $P(B_2) = 1$ for

$$B_2 := \{\text{for each } n \geq 1, \text{ there exists } U_n \geq t \text{ for any } t > 0 \text{ such that } B_{U_n} \leq -r_2^{(n)}(U_n)\}.$$

Under the event $A \cap B_2$, since there exists $T_3 \geq t \vee T_1$ for any $t > 0$ and $N \geq 1$ in (2.5) such that

$$B_{T_3} \leq -r_2^{(N)}(T_3) = -\sqrt{T_3}g(T_3) - N,$$

we have for such T_3,

$$R_{T_3} \leq \frac{d-1}{2}T_3 - \sqrt{T_3}g(T_3)$$

by (2.6). Therefore, the probability in (1.3) is 0. □

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