On the Letac-Massam Conjecture on cones Q_{A_n}

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Abstract: We prove, for graphical models for nearest neighbour interactions, a conjecture stated by Letac and Massam in 2007. Our result is important in the analysis of Wishart distributions on cones related to graphical models and in its statistical applications.

Key words: Laplace transform; power function; analysis on cones; Wishart distribution; graphical model.

1. Preliminaries.

1.1. Introduction. In this note, we solve on an important class of cones, the conjecture stated by Letac and Massam in [7, p. 1314], and called "Letac-Massam Conjecture" in [1]. This conjecture is of fundamental importance in harmonic analysis of Riesz and Wishart measures on convex cones connected to graphs and in its applications to modern multivariate statistics. More generally, the Conjecture of Letac-Massam is closely related to an important problem in a wide range of analysis on cones:

($\mathcal{P}$) *Is the Laplace transform of a product of powers of given polynomials equal to a product of powers of some polynomials*?

Let $n \geq 2$. We denote by $\mathrm{Sym}(n, \mathbf{R})$ the vector space of real symmetric $n \times n$ matrices and by $\mathrm{Sym}^+(n, \mathbf{R})$ the cone of symmetric positive definite matrices. Let G be a graph with vertices $1, 2, \ldots, n$ and edges E. Let Z_G be the subspace of $\mathrm{Sym}(n, \mathbf{R})$ containing matrices z with $z_{ij} = 0$ if $\{i, j\} \notin E$. Cones $P_G = \mathrm{Sym}^+(n, \mathbf{R}) \cap Z_G$ and their dual cones Q_G are basic objects of graphical model theory ([5], [7]), one of the most important parts of contemporary statistics, including big data statistics. We refer to [5] and [7] for all the notions and notations not explained in detail in this paper.

We show that the Letac-Massam Conjecture is true on the cones Q_G where

$$G = A_n = 1 - 2 - \cdots - n.$$

This is a fundamental class of non-homogeneous cones appearing in the statistical theory of graphical models [5], corresponding to the practical model of nearest neighbour interactions. In the Gaussian character $(X_1, X_2, \ldots, X_n)$, non-neighbours X_i, X_j, $|i - j| > 1$ are conditionally independent with respect to other variables.

According to [7, Corollary 3.1], the Letac-Massam Conjecture is true on the cones Q_{A_4} and P_{A_4}, but these results are "obtained by a nontrivial and long computation" and the proofs are omitted. [7] states that for $n = 5$ "calculations are terrifying." Our method of proof is simple and based on methods of [4]: triangular changes of variables on Q_{A_n} and using natural "future" and "past" power functions $\delta_{\underline{s}}^{(M)}$ and $\Delta_{\underline{s}}^{(M)}$ on Q_{A_n} and on P_{A_n}. Our method also applies to the cone P_{A_4}.

1.2. Letac-Massam Conjecture. The Letac-Massam Conjecture is a conjecture on the Laplace transform of functions $\eta \mapsto H(\alpha, \beta, \eta)$, $\eta \in Q_{A_n}$, $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$, $\beta = (\beta_2, \ldots, \beta_{n-1})$, introduced in [7], cf. (2) below. If needed, we will use a more precise notation H_n for the function H on Q_{A_n}. Let π denote the projection from $\mathrm{Sym}^+(n, \mathbf{R})$ onto the cone Q_{A_n} and let $\mu_{A_n}(d\eta)$ be the reference measure on the cone Q_{A_n}, defined by (3) below. The Letac-Massam Conjecture on the cone Q_{A_n} says that there exists $C_{\alpha,\beta} > 0$ such that

$$(1) \qquad \int_{Q_{A_n}} e^{-\mathrm{tr}(y\eta)} H(\alpha, \beta, \eta) d\mu_{A_n}(\eta) = C_{\alpha,\beta} H(\alpha, \beta, \pi(y^{-1})) \qquad (y \in P_{A_n})$$

if and only if $(\alpha, \beta) \in \mathcal{A}$, where $\mathcal{A} = \bigcup_{M=2}^{n-1} A_M$ and the sets A_M are defined by the following conditions (C) and (I):

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(C) $\alpha_j = \beta_{j+1}$ if $1 \le j \le M-2$, and $\alpha_j = \beta_j$ if $M+1 \le j \le n-1$,

(I) $\alpha_j > \frac{1}{2}$ for all $j = 1, \ldots, n-1$, and $\alpha_{M-1} + \alpha_M - \beta_M > 0$.

The sufficiency of the condition $(\alpha, \beta) \in \mathcal{A}$ was showed in [7] and the necessity was conjectured and announced true for $n = 4$. For $n = 2$ and $n = 3$ the equivalence of (1) with $(\alpha, \beta) \in \mathcal{A}$ is well known. The necessity of (I) is evident (consider diagonal $y \in P_{A_n}$, cf. Lemma 2.1 below), so the necessity of (C) is to be proved for $n \ge 4$.

In the sequel, the equality (1) will be referred to as the Letac-Massam formula on Q_{A_n} and the conditions (C) as Letac-Massam conditions. In this note we prove

Theorem 1.1. *Let $n \ge 4$. The formula* (1) *implies conditions* (C).

1.3. Letac-Massam Conjecture for power functions $\delta_{\underline{s}}^{(M)}$ and $\Delta_{\underline{s}}^{(M)}$. Now we introduce the power functions $\delta_{\underline{s}}^{(M)}$ on Q_{A_n} and $\Delta_{\underline{s}}^{(M)}$ on P_{A_n}. For all $2 \le M \le n-1$,

$$\delta_{\underline{s}}^{(M)}(\eta) = \frac{\prod_{i=1}^{M-1} |\eta_{\{i:i+1\}}|^{s_i} \prod_{i=M+1}^{n} |\eta_{\{i-1:i\}}|^{s_i}}{\prod_{i=2}^{M-1} \eta_{ii}^{s_{i-1}} \eta_{MM}^{s_{M-1}-s_M+s_{M+1}} \prod_{i=M+1}^{n-1} \eta_{ii}^{s_{i+1}}}$$

$$\Delta_{\underline{s}}^{(M)}(y) = \prod_{i<M} |y_{\{1:i\}}|^{s_i - s_{i+1}} |y|^{s_M} \prod_{i>M} |y_{\{i:n\}}|^{s_i - s_{i-1}},$$

where, for $I \subset \{1, \ldots, n\}$, the matrix A_I is the submatrix of A indexed by I, and the symbol $\{a : b\}$ with $1 \le a \le b \le r$ denotes the set of i for which $a \le i \le b$. The relation $\delta_{\underline{s}}^{(M)}(\pi(y^{-1})) = \Delta_{-\underline{s}}^{(M)}(y)$ is proved in [4]. Define $r_i = \alpha_{i+1} - \beta_i$ and $p_i = \alpha_i - \beta_i$ for all $2 \le i \le n-1$. We have, as defined in [7],

$$H(\alpha, \beta, \eta) = \frac{\prod_{i=1}^{n-1} |\eta_{\{i:i+1\}}|^{\alpha_i}}{\prod_{i=2}^{n-1} \eta_{ii}^{\beta_i}} \tag{2}$$

so that $H(\alpha, \beta, \eta) = \delta_{\underline{s}}^{(M)}(\eta) \prod_{i=2}^{M-1} \eta_{ii}^{r_i} \prod_{i=M+1}^{n-1} \eta_{ii}^{p_i}$, where $s_i = \alpha_i$, for all $1 \le i \le M-1$; $s_i = \alpha_{i-1}$, for all $M+1 \le i \le n$ and $s_M = \alpha_{M-1} + \alpha_M - \beta_M$. This implies $r_M = s_M - s_{M+1}$ and $p_M = s_M - s_{M-1}$. We notice that $\underline{s} = (s_i)$ depends on M, whereas neither $\underline{r} = (r_i)$ nor $\underline{p} = (p_i)$ does.

Let $\varphi(y) = \pi(y^{-1})$ and

$$\mu_{A_n}(d\eta) = \prod_{i=1}^{n-1} |\eta_{\{i,i+1\}}|^{-3/2} \prod_{i \ne 1,n} \eta_{ii} d\eta \tag{3}$$

be the reference measure used in [7], where $d\eta$ is the Lebesgue measure.

The Letac-Massam formula (1) is equivalent, for each $2 \le M \le n-1$, to

$$\int_{Q_{A_n}} e^{-\mathrm{tr}(y\eta)} \delta_{\underline{s}}^{(M)}(\eta) \prod_{i=2}^{M-1} \eta_{ii}^{r_i} \prod_{i=M+1}^{n-1} \eta_{ii}^{p_i} d\mu_{A_n}(\eta) = C_{\alpha,\beta} \Delta_{-\underline{s}}^{(M)}(y) \prod_{i=2}^{M-1} \varphi(y)_{ii}^{r_i} \prod_{i=M+1}^{n-1} \varphi(y)_{ii}^{p_i}. \tag{4}$$

The Letac-Massam conditions (C) are equivalent to the following $n-2$ alternative conditions:

$$r_2 = \cdots = r_{M-1} = p_{M+1} = \cdots = p_{n-1} = 0 \tag{5}$$

for an $M \in \{2, \ldots, n-1\}$, or, in other words, to the equality $H(\alpha, \beta, \cdot) = \delta_{\underline{s}}^{(M)}$ for an $M \in \{2, \ldots, n-1\}$.

A positive answer to the Letac-Massam Conjecture implies that the natural generalized power functions on Q_{A_n} are the families $\delta_{\underline{s}}^{(M)}(\eta)$, motivated by analysis on symmetric and homogeneous cones, with n-dimensional parameter $\underline{s}$. Power functions $H(\alpha, \beta, \eta)$, $\eta \in Q_{A_n}$ are motivated by advanced graph theory, more exactly by cliques and separators formalism. The parameters α, β have dimension $2n-3$. Even if we start with a larger family $H(\alpha, \beta, \eta)$, in order to have the property $(\mathcal{P})$ satisfied, we boil down to the families $\delta_{\underline{s}}^{(M)}(\eta)$, with $M = 2, \ldots, n-1$. Moreover, the families $\delta_{\underline{s}}^{(1)}(\eta)$ and $\delta_{\underline{s}}^{(n)}(\eta)$ are "forgotten" in the graph theory approach of [7].

2. Proof. We are going to prove the Letac-Massam Conjecture by induction on n. The proof of the initiation part ($n = 4$) and the heredity part ($n \ge 5$) are the same, so they are given together. We use extensively the bijections Ψ_n and $\tilde{\Psi}_n$ between $\mathbf{R}^+ \times \mathbf{R} \times Q_{A_{n-1}}$ and Q_{A_n} and the bijections Φ_n and $\tilde{\Phi}_n$ between $\mathbf{R}^+ \times \mathbf{R} \times P_{A_{n-1}}$ and P_{A_n}, studied in [4] (see Appendix for more information).

First, in the following lemma, we express, for each M, the constant $C_{\alpha,\beta}$ as a function of $M, \underline{s} = (s_i), \underline{r} = (r_i)$ and $\underline{p} = (p_i)$. This is convenient and needed in further study of the formula (4).

Lemma 2.1. *If the formula* (4) *holds for all $y \in P_{A_n}$ then we have*

$$C_{\alpha,\beta} = \pi^{(n-1)/2} \Gamma(s_M) \left\{ \prod_{i \ne M} \Gamma\left(s_i - \frac{1}{2}\right) \right\} \times \prod_{i=2}^{M-1} \frac{\Gamma(s_i + r_i)}{\Gamma(s_i)} \prod_{i=M+1}^{n-1} \frac{\Gamma(s_i + p_i)}{\Gamma(s_i)}. \tag{6}$$

If y is diagonal, then (4) *holds if and only if $s_i > \frac{1}{2}$ for $i \ne M$, $s_M > 0$, $s_i + r_i > 0$ for $2 \le i < M$ and $s_i + p_i > 0$ for $M < i \le n-1$.*

Proof. We take y diagonal. The proof is a by-product of the Step 1 of the main proof. □

Step 1 (descent in Letac-Massam formula, from Q_{A_n} to $Q_{A_{n-1}}$)**.** *Let* $n \geq 4$, $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$ *and* $\beta = (\beta_2, \ldots, \beta_{n-1})$.

Suppose that the Letac-Massam formula (1) *holds for* $H_n(\alpha, \beta, \cdot)$ *on* Q_{A_n}*. Then the Letac-Massam formula holds on* $Q_{A_{n-1}}$ *for*:
(i) $H_{n-1}((\alpha_1, \ldots, \alpha_{n-2}), (\beta_2, \ldots, \beta_{n-2}), \cdot)$ *and the graph* $1 - \cdots - (n-1)$,
(ii) $H_{n-1}((\alpha_2, \ldots, \alpha_{n-1}), (\beta_3, \ldots, \beta_{n-1}), \cdot)$ *and the graph* $2 - \cdots - n$.

Proof. Let us prove (i). We choose $2 \leq M \leq n-2$. For all $y \in P_{A_n}$, let, successively, $y = \tilde{\Phi}_n(a', b', z)$ and $z = \Phi_{n-1}(a'', b'', Z)$. We easily check that for $2 \leq i \leq n-1$, $\varphi(y)_{ii} = \varphi(z)_{ii} = \varphi(Z)_{ii}$, see Lemma 4.2 (by our convention, z is indexed by $1, \ldots, n-1$ and Z is indexed by $2, \ldots, n-1$). Integration on Q_{A_n} with two successive changes of variables $\eta = \tilde{\Psi}_n(\mu', \nu', \xi)$ and then $\xi = \Psi_{n-1}(\mu'', \nu'', \Xi)$ gives

$$(7) \quad \int_{Q_{A_{n-2}}} e^{-\mathrm{tr}(Z\Xi)} \delta^{(M)}_{(s_2,\ldots,s_{n-1})}(\Xi) \prod_{i=2}^{M-1} \Xi_{ii}^{r_i} \times \prod_{i=M+1}^{n-1} \Xi_{ii}^{p_i} d\mu_{A_{n-2}}(\Xi) = C^{(n-2)}_{\alpha,\beta} \Delta^{(M)}_{-(s_2,\ldots,s_{n-1})}(Z) \prod_{i=2}^{M-1} \varphi(Z)_{ii}^{r_i} \prod_{i=M+1}^{n-1} \varphi(Z)_{ii}^{p_i},$$

where $C^{(n-2)}_{\alpha,\beta} = \frac{C_{\alpha,\beta}}{\pi \Gamma(s_1 - \frac{1}{2}) \Gamma(s_n - \frac{1}{2})}$ and the rows and columns of Ξ and Z are numbered $2, \ldots, n-1$. Now, we apply one more change of variable $\Xi = \tilde{\Psi}_{n-2}(\mu, \nu, \Theta)$ in formula (7) and we set $Z = \tilde{\Phi}_{n-2}(a, 0, T)$. The lines and columns of Θ and T are numbered $2, \ldots, n-2$. Let $F(\mu, \nu, \Theta)$ be the integrand. We first compute $J = \int_{-\infty}^{\infty} \int_0^{\infty} F d\mu d\nu = 2 \int_0^{\infty} \int_0^{\infty} F d\mu d\nu$. Using the change of variables $u = a\mu$, $t = a\Theta_{n-2,n-2}\nu^2$ we get

$$J = 2a^{-p_{n-1}} \int_0^{\infty} \int_0^{\infty} e^{-\left(a\mu + a\Theta_{n-2,n-2}\nu^2\right)} \times \mu^{s_{n-1}-3/2} (a\mu + a\Theta_{n-2,n-2}\nu^2)^{p_{n-1}} d\mu d\nu$$
$$= a^{-(s_{n-1}+p_{n-1})} \Theta^{-1/2}_{n-2,n-2} \times \int_0^{\infty} \int_0^{\infty} e^{-(u+t)} u^{s_{n-1}-\frac{3}{2}} t^{-\frac{1}{2}} (u+t)^{p_{n-1}} du dt.$$

Now, using the change of variables $(u, v) = (u, u+t)$, we get

$$(8) \quad J = a^{-(s_{n-1}+p_{n-1})} \Theta^{-1/2}_{n-2,n-2} \times \int_0^{\infty} \left(\int_0^{v} u^{s_{n-1}-\frac{3}{2}} (v-u)^{-\frac{1}{2}} du \right) e^{-v} v^{p_{n-1}} dv$$
$$= a^{-(s_{n-1}+p_{n-1})} \Theta^{-1/2}_{n-2,n-2} \times B\left(s_{n-1} - \frac{1}{2}, \frac{1}{2}\right) \Gamma(s_{n-1} + p_{n-1}),$$

where, in the integral with respect to du we made a change of variable $x = u/v$. We get

$$(9) \quad \int_{Q_{A_{n-3}}} e^{-\mathrm{tr}(T\Theta)} \delta^{(M)}_{(s_2,\ldots,s_{n-2})}(\Theta) \times \prod_{i=2}^{M-1} \Theta_{ii}^{r_i} \prod_{i=M+1}^{n-2} \Theta_{ii}^{p_i} d\mu_{A_{n-3}}(\Theta)$$
$$= C^{(n-3)}_{\alpha,\beta} \Delta^{(M)}_{-(s_2,\ldots,s_{n-2})}(T) \times \prod_{i=2}^{M-1} \varphi(T)_{ii}^{r_i} \prod_{i=M+1}^{n-2} \varphi(T)_{ii}^{p_i},$$

where

$$(10) \quad C^{(n-3)}_{\alpha,\beta} = \frac{C_{\alpha,\beta}}{\pi^{\frac{3}{2}} \Gamma(s_1 - \frac{1}{2}) \Gamma(s_n - \frac{1}{2}) \Gamma(s_{n-1} - \frac{1}{2})} \times \frac{\Gamma(s_{n-1})}{\Gamma(p_{n-1} + s_{n-1})}.$$

Recall that throughout the paper $C_{\alpha,\beta}$ denotes the constant from formulas (1) and (4). By the same argument as to obtain formula (7), we observe that the Letac-Massam formula for the function $H_{n-1}((\alpha_1, \ldots, \alpha_{n-2}), (\beta_2, \ldots, \beta_{n-2}), \cdot)$ on $Q_{A_{n-1}}$ and the graph $1 - 2 - \cdots - (n-1)$ is equivalent to formula (9). This finishes the proof of (i).

By a similar "mirror-like" argument with the change of variables $\Xi = \Psi_{n-2}(\mu, \nu, \Theta)$ in (7), we get the Letac-Massam formula for $H_{n-1}((\alpha_2, \ldots, \alpha_{n-1}), (\beta_3, \ldots, \beta_{n-1}), \cdot)$ and the graph $2 - \cdots - n$, and we prove part (ii) of Step 1. □

Proof of Lemma 2.1. For y diagonal, formula (10) leads by induction to formula (6), observing that the last equation we get is $a^{-s_M} \int_0^{\infty} e^{-ax} x^{s_M} \frac{dx}{x} = C^{(1)}_{\alpha,\beta} a^{-s_M}$, so that $C^{(1)}_{\alpha,\beta} = \Gamma(s_M)$. □

Step 2 (induction step)**.** *The Letac-Massam Conjecture on* $Q_{A_{n-1}}$ *implies the Letac-Massam Conjecture on* Q_{A_n}.

Proof. Let $n \geq 4$. Suppose that the Letac-Massam formula (1) holds for some α and β and suppose that the Letac-Massam Conjecture is true on $Q_{A_{n-1}}$.

For $n \geq 5$, we use Step 1 and the induction hypothesis. Thus one of the following $n-3$ conditions has to be satisfied:

$$r_2 = \cdots = r_{M-1} = p_{M+1} = \cdots = p_{n-2} = 0,$$

for an $M \in \{2, \cdots, n-2\}$, and, simultaneously, one of the following $n-3$ "shifted" conditions has to be satisfied:

$$r_3 = \cdots = r_M = p_{M+2} = \cdots = p_{n-1} = 0,$$

for an $M \in \{2, \ldots, n-2\}$. This implies that either conditions (5) are satisfied or

$$\text{(11)} \qquad p_3 = \cdots = p_{n-2} = 0; \; r_3 = \cdots = r_{n-2} = 0.$$

Let us assume this single remaining case and show that it also implies conditions (5).

The equality $r_M = 0$ implies $s_M = s_{M+1}$ and $p_M = 0$ implies $s_M = s_{M-1}$. Also, from $p_j = r_j$ for all $3 \leq j \leq n-2$, we get $s_2 = \cdots = s_{M-1}$ and $s_{M+1} = \cdots = s_{n-1}$. Thus, $s_2 = \cdots = s_{n-1} = s$. In the case (11), using the cofactor formula for Z^{-1}, equation (7) reduces to

$$\text{(12)} \quad \int_{Q_{A_{n-2}}} e^{-\mathrm{tr}(Z\Xi)} \delta^{(M)}_{(s,\ldots,s)}(\Xi) \Xi_{22}^{r_2} \Xi_{n-1,\,n-1}^{p_{n-1}} d\mu_{A_{n-2}}(\Xi) = C^{(n-2)}_{\alpha,\beta} |Z|^{-s} \left(\frac{|Z_{\{3:n-1\}}|}{|Z|} \right)^{r_2} \left(\frac{|Z_{\{2:n-2\}}|}{|Z|} \right)^{p_{n-1}}.$$

We apply the second derivative with respect to $Z_{n-2,n-1}$ on both sides of (12) and we take $Z_{n-2,n-1} = 0$. Theorem 2.7.1 in [6] ensures that the derivatives of all orders of the integral (12) can be computed under the integral sign. We obtain

$$\text{(13)} \qquad \int_{Q_{A_{n-2}}} e^{-\mathrm{tr}(Z\Xi)} \delta^{(M)}_{(s,\ldots,s)}(\Xi) \Xi_{22}^{r_2} \Xi_{n-1,\,n-1}^{p_{n-1}} \times \Xi^2_{n-2,n-1} d\mu_{A_{n-2}}(\Xi)|_{Z_{n-2,n-1}=0} = \frac{C^{(n-2)}_{\alpha,\beta}}{4} \frac{\partial^2}{\partial Z^2_{n-2,n-1}} \bigg|_{Z_{n-2,n-1}=0} g(Z),$$

where $g(Z) = |Z|^{-s} \left(\frac{|Z_{\{3:n-1\}}|}{|Z|} \right)^{r_2} \left(\frac{|Z_{\{2:n-2\}}|}{|Z|} \right)^{p_{n-1}}$.

Let us change the variables $\Xi = \tilde{\Psi}_{n-2}(\tilde{\mu}, \tilde{\nu}, \Theta)$ and set $Z = \tilde{\Phi}_{n-2}(a, 0, T)$, i.e. $Z_{n-2,n-1} = 0$. Similarly as in the proof of (9) in Step 1, we find that the left hand side of (13) is

$$\text{(14)} \quad a^{-(s+p_{n-1}+1)} \Gamma(s + p_{n-1} + 1) B\left(s - \tfrac{1}{2}, \tfrac{3}{2}\right) \times \int_{Q_{A_{n-3}}} e^{-\mathrm{tr}(T\Theta)} \delta^{(M)}_{(s,\ldots,s)}(\Theta) \Theta_{22}^{r_2} \Theta_{n-2,n-2} d\mu_{An-3}(\Theta).$$

We write $\frac{1}{4} C^{(n-2)}_{\alpha,\beta} D$ the right hand side of (13) and we compute D. Denoting $S = -(s + r_2 + p_{n-1})$ and $h(Z) = |Z|^S |Z_{\{3:n-1\}}|^{r_2}$ we have

$$D = |Z_{\{2:n-2\}}|^{p_{n-1}} \frac{\partial^2}{\partial Z^2_{n-2,n-1}} \bigg|_{Z_{n-2,n-1}=0} h(Z).$$

We apply formulas

$$|Z| = Z_{n-1,n-1} |Z_{\{2:n-2\}}| - Z^2_{n-2,n-1} |Z_{\{2:n-3\}}|,$$

$$|Z|^S = \left(Z_{n-1,n-1} |Z_{\{2:n-2\}}|\right)^S \times \left(1 - S \frac{Z^2_{n-2,n-1} |Z_{\{2:n-3\}}|}{Z_{n-1,n-1} |Z_{\{2:n-2\}}|} + o(Z^2_{n-2,n-1}) \right).$$

Thus, for $Z_{n-2,n-1} = 0$, we get $\frac{\partial |Z|^S}{\partial Z_{n-2,n-1}} = 0$ and $\frac{\partial^2 |Z|^S}{\partial Z^2_{n-2,n-1}} = -2S \left(Z_{n-1,n-1} |Z_{\{2:n-2\}}|\right)^{S-1} |Z_{\{2:n-3\}}|$.

Similarly, $|Z_{\{3:n-1\}}|$

$$= Z_{n-1,n-1} |Z_{\{3:n-2\}}| - Z^2_{n-2,n-1} |Z_{\{3:n-3\}}|$$

(for $n = 5$ we set $|Z_{\{3:n-3\}}| = 1$) and

$$\frac{\partial^2 |Z_{\{3:n-1\}}|^{r_2}}{\partial Z^2_{n-2,n-1}} \bigg|_{Z_{n-2,n-1}=0} = -2r_2 \left(Z_{n-1,n-1} |Z_{\{3:n-2\}}|\right)^{r_2 - 1} |Z_{\{3:n-3\}}|.$$

Using $Z = \tilde{\Phi}_{n-2}(a, 0, T)$, where the matrix T is indexed by $2, \ldots, n-2$, we obtain $Z_{n-1,n-1} = a$, $Z_{\{2:n-2\}} = T$, $Z_{\{3:n-2\}} = T_{\{3:n-2\}}$, $|Z_{\{3:n-1\}}| = a|T_{\{3:n-2\}}|$ and $|Z| = a|T|$. By Leibniz formula,

$$D = -2a^{r_2+S-1} |T|^{p_{n-1}+S-1} |T_{\{3:n-2\}}|^{r_2-1} \times \left(S |T_{\{3:n-2\}}| |T_{\{2:n-3\}}| + r_2 |T_{\{3:n-3\}}| |T| \right),$$

where for $n = 5$ we set $|T_{\{3:n-3\}}| = 1$. Hence, for $Z_{n-2,n-1} = 0$, the right hand side of (13) is

$$\text{(15)} \quad \frac{C^{(n-2)}_{\alpha,\beta}}{2} a^{-(s+p_{n-1}+1)} |T|^{-(s+r_2+1)} |T_{\{3:n-2\}}|^{r_2 - 1} f(T),$$

where

$$f(T) = (s + r_2 + p_{n-1}) |T_{\{3:n-2\}}| |T_{\{2:n-3\}}| - r_2 |T_{\{3:n-3\}}| |T|.$$

Equating (14) and (15), we obtain, using (10),

$$\text{(16)} \qquad \int_{Q_{A_{n-3}}} e^{-\mathrm{tr}(T\Theta)} \delta^{(M)}_{(s,\ldots,s)} \Theta_{22}^{r_2} \Theta_{n-2,n-2} d\mu_{A_{n-3}}(\Theta) = \frac{sd(s, r_2, T)}{s + p_{n-1}} f(T),$$

where

$$d(s, r_2, T) = C^{(n-3)}_{\alpha,\beta} |T|^{-(s+r_2+1)} |T_{\{3:n-2\}}|^{r_2 - 1}.$$

Formula (16) is supposed to be true for our $p_{n-1} = \alpha_{n-1} - \beta_{n-1}$. It is surely true for $p_{n-1} = 0$, because the Letac-Massam conditions (5) are then satisfied. Equating (16) for these two values of p_{n-1}, and noting that by (6) the constant $C^{(n-3)}_{\alpha,\beta}$ does not depend on p_{n-1}, we get

$$\frac{(s+r_2+p_{n-1})|T_{\{3:n-2\}}||T_{\{2:n-3\}}|-r_2|T_{\{3:n-3\}}||T|}{s+p_{n-1}} = \frac{(s+r_2)|T_{\{3:n-2\}}||T_{\{2:n-3\}}|-r_2|T_{\{3:n-3\}}||T|}{s},$$

which is equivalent to

$$(17)\qquad r_2p_{n-1}\big(|T_{\{3:n-2\}}||T_{\{2:n-3\}}|-|T_{\{3:n-3\}}||T|\big)=0,$$

where for $n=5$ we set $|T_{\{3:n-3\}}|=1$. We observe that $|T_{\{3:n-2\}}||T_{\{2:n-3\}}|-|T_{\{3:n-3\}}||T|\neq 0$, for example for T such that $T_{ii}=2$ for all $2\le i\le n-2$, $T_{i,i+1}=T_{i+1,i}=1$ for $2\le i\le n-3$ and $T_{ij}=0$ for all other i,j (in this case, this expression equals 1). Thus, for $n\ge 5$, in the remaining case (11), we also have $r_2=0$ or $p_{n-1}=0$. In both cases we fall in the Letac-Massam conditions (5) and the proof of the induction step is finished.

For $n=4$, we get formula (7) for $M=2$, the computations are simpler (no use of Leibniz formula is needed), and no condition $s_2=s_3=s$ appears. The analogue of formula (16) is

$$(18)\quad \Gamma(s_3+p_3+1)B\bigg(s_3-\frac{1}{2},\frac{3}{2}\bigg)\int_0^\infty e^{-tu}u^{s_2}u\frac{1}{u}\,du = \frac{C^{(2)}_{\alpha,\beta}}{2}(s_2+p_3)t^{-(s_2+1)},\quad t>0.$$

After substitution of the constant

$$C^{(2)}_{\alpha,\beta}=\pi^{\frac{1}{2}}\Gamma(s_2)\Gamma\bigg(s_3-\frac{1}{2}\bigg)\frac{\Gamma(s_3+p_3)}{\Gamma(s_3)},$$

one gets $(s_3+p_3)s_2=s_3(s_2+p_3)$ equivalent to $r_2p_3=0$, so $r_2=0$ or $p_3=0$. We get the Letac-Massam conditions for Q_{A_4}. □

Remark 2.2. The expression on the RHS of (17), i.e. $|T_{\{3:n-2\}}||T_{\{2:n-3\}}|-|T_{\{3:n-3\}}||T|$, where $T=T_{\{2:n-2\}}$ is known in matrix theory. It is treated in Desnanot-Jacobi identity ([2, Thm 3.12]), called also Lewis Caroll (or Dodgson's) identity ([3]) and is equal to $(\prod_{i=2}^{n-3}T_{i,i+1})^2$, the square of the monomial of the off-diagonal entries.

Remark 2.3. The same method applies in order to prove the Letac-Massam Conjecture on P_{A_4}. We take $M=2$ and apply two changes of variables Φ_4 and $\tilde\Phi_3$ on P_{A_4} and P_{A_3}, see Lemma 4.1. We obtain an integral on $P_{A_2}=\mathrm{Sym}^+(2,\mathbf{R})$, which is the same as the integral on $Q_{A_2}=\mathrm{Sym}^+(2,\mathbf{R})$ in the proof above. The work on the Letac-Massam Conjecture on P_{A_n} for $n\ge 5$ is in progress. The analysis on these cones is more difficult.

Remark 2.4. Our method of differentiating the Letac-Massam formula with respect to Z_{12} gives a simple proof of the "Mellin transform" Lemma 3.1 in [7, p. 1302], announced without proof. However, instead of the second derivative in Z_{12}, the complete Taylor expansion in Z_{12} is needed.

Remark 2.5. Sufficiency of Letac-Massam conditions follows from Gamma-Siegel integrals, i.e. formulas for the Laplace transform of $\delta^{(M)}_{\underline{s}}$ and $\Delta^{(M)}_{\underline{s}}$, proved in [4], using the triangular changes of variables from Lemma 4.1.

3. Generalized Letac-Massam Conjecture. In the first part of the proof of Theorem 1.1, we showed that the Letac-Massam formula (1) on Q_{A_n}, with $M=2,\dots,n-1$, is equivalent to a Laplace transform formula (4) on $Q_{A_{n-2}}$, for a function $\delta^{(M)}_{(s_2,\dots,s_{n-1})}$. Next we proved that (4) implies that the formula is rewritten for an $M'\in\{2,\dots,n-1\}$ with $r_i=0=p_j, i=2,\dots,M'-1, j=M'+1,\dots,n-1$. Thus, in fact we showed a stronger result that we call Generalized Letac-Massam Conjecture (GLMC):

Theorem 3.1. *Let $M\in\{1,\dots,n\}$. There exist a multi-index $\underline{s}\in\mathbf{R}^n$ and a constant $C>0$ such that for all $y\in P_{A_n}$*

$$\int_{Q_{A_n}} e^{-\mathrm{tr}(y\eta)}\delta^{(M)}_{\underline{s}}(\eta)\prod_{i=1}^{M-1}\eta_{ii}^{r_i}\prod_{i=M+1}^{n}\eta_{ii}^{p_i}\,d\mu_{A_n}(\eta) = C\,\Delta^{(M)}_{-\underline{s}}(y)\prod_{i=1}^{M-1}\varphi(y)_{ii}^{r_i}\prod_{i=M+1}^{n}\varphi(y)_{ii}^{p_i}$$

if and only if the formula is rewritten with $M'\in\{1,\dots,n\}$ such that $r_i=0=p_j, i=1,\dots,M'-1, j=M'+1,\dots,n$ and $s_i>\frac{1}{2}, i\neq M', s_{M'}>0$.

The GLMC gives a partial answer to the question which products of powers of well-defined minors on Q_{A_n} have the property $(\mathcal{P})$.

4. Appendix.

4.1. Changes of variables.

Lemma 4.1. *Let $n\ge 2$.*

(i) *Let* $\Phi_n:\mathbf{R}^+\times\mathbf{R}\times P_{A_{n-1}}\longrightarrow P_{A_n}$, $(a,b,z)\longmapsto y$ *with*

$$y=A(b)\begin{pmatrix} a & 0 & \cdots & 0\\ 0 & & & \\ \vdots & & z & \\ 0 & & & \end{pmatrix}{}^{\mathrm{t}}A(b),$$

$$A(b) = \begin{pmatrix} 1 & & & \\ b & 1 & & \\ \vdots & & \ddots & \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

and let $\Psi_n : \mathbf{R}^+ \times \mathbf{R} \times Q_{A_{n-1}} \longrightarrow Q_{A_n}$, $(\mu, \nu, \xi) \longmapsto \eta$ *with*

$$\eta = \pi \left({}^{\mathrm{t}}A(\nu) \begin{pmatrix} \mu & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \xi & \\ 0 & & & \end{pmatrix} A(\nu) \right).$$

Then the maps Φ_n *and* Ψ_n *are bijections.*

(ii) *Let* $\tilde{\Phi}_n : \mathbf{R}^+ \times \mathbf{R} \times P_{A_{n-1}} \longrightarrow P_{A_n}$, $(a, b, z) \longmapsto \tilde{y}$ *with*

$$\tilde{y} = {}^{\mathrm{t}}B(b) \begin{pmatrix} & & & 0 \\ & z & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & a \end{pmatrix} B(b),$$

$$B(b) = \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & \dots & b & 1 \end{pmatrix},$$

and let $\tilde{\Psi}_n : \mathbf{R}^+ \times \mathbf{R} \times Q_{A_{n-1}} \longrightarrow Q_{A_n}$, $(\mu, \nu, \xi) \longmapsto \tilde{\eta}$ *with*

$$\tilde{\eta} = \pi \left(B(\nu) \begin{pmatrix} & & & 0 \\ & \xi & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & \mu \end{pmatrix} {}^{\mathrm{t}}B(\nu) \right).$$

Then the maps $\tilde{\Phi}_n$ *and* $\tilde{\Psi}_n$ *are bijections.*

(iii) *Let* $y = \Phi_n(a, b, z)$ *and* $\eta = \Psi_n(\mu, \nu, \xi)$. *Then, for all* $M = 2, \dots, n$,

$$\Delta_{\underline{s}}^{(M)}(y) = a^{s_1} \Delta_{(s_2,\dots,s_n)}^{(M)}(z),$$
$$\delta_{\underline{s}}^{(M)}(\eta) = \mu^{s_1} \delta_{(s_2,\dots,s_n)}^{(M)}(\xi).$$

Let $y = \tilde{\Phi}_n(a, b, z)$ *and* $\eta = \tilde{\Psi}_n(\mu, \nu, \xi)$. *Then, for all* $M = 1, \dots, n-1$,

$$\Delta_{\underline{s}}^{(M)}(y) = a^{s_n} \Delta_{(s_1,\dots,s_{n-1})}^{(M)}(z),$$
$$\delta_{\underline{s}}^{(M)}(\eta) = \mu^{s_n} \delta_{(s_1,\dots,s_{n-1})}^{(M)}(\xi).$$

(iv) *If* $y = \Phi_n(a, b, z)$ *and* $\eta = \Psi_n(\mu, \nu, \xi)$, *then*

$$\mathrm{tr}(y\eta) = a\mu + a\xi_{22}(b + \nu)^2 + \mathrm{tr}(z\xi).$$

If $y = \tilde{\Phi}_n(a, b, z)$ *and* $\eta = \tilde{\Psi}_n(\mu, \nu, \xi)$, *then*

$$\mathrm{tr}(y\eta) = a\mu + a\xi_{n-1,n-1}(b + \nu)^2 + \mathrm{tr}(z\xi).$$

(v) *The Jacobian of the change of variables* $\eta = \Psi_n(\mu, \nu, \xi)$ *on* Q_{A_n} *is* $J_{\Psi_n}(\mu, \nu, \xi) = \xi_{22}$.

The Jacobian of the change of variables $\eta = \tilde{\Psi}_n(\mu, \nu, \xi)$ *on* Q_{A_n} *is* $J_{\tilde{\Psi}_n}(\mu, \nu, \xi) = \xi_{n-1,n-1}$.

The Jacobians of the changes of variables $y = \Phi_n(a, b, z)$ *or* $y = \tilde{\Phi}_n(a, b, z)$ *on* P_{A_n} *are equal to* a.

Lemma 4.2. *Consider* $y \in P_{A_n}$.

(i) *If* $y = \Phi_n(a, b, z)$, *then* $\varphi(z)_{jj} = \varphi(y)_{jj}$ *for* $j \geq 2$.

(ii) *If* $y = \tilde{\Phi}_n(a, b, z)$, *then* $\varphi(z)_{jj} = \varphi(y)_{jj}$ *for* $j \leq n - 1$.

Proof. (i) Note that $y = \Phi_n(a, b, z)$ is expressed in the form $T\begin{pmatrix} a & \\ & z \end{pmatrix}{}^{\mathrm{t}}T$, where $T = A(b)$ in Lemma 4.1. In general, let M, R, S be $n \times n$ matrices with R upper triangular and S lower triangular. Then $(RMS)_{\{j:n\}} = R_{\{j:n\}} M_{\{j:n\}} S_{\{j:n\}}$. It follows that $(y^{-1})_{\{j:n\}} = (({}^{\mathrm{t}}T)^{-1})_{\{j:n\}} (z^{-1})_{\{j:n\}} (T^{-1})_{\{j:n\}} = (z^{-1})_{\{j:n\}}$ for $j \geq 2$ since $(T^{-1})_{\{j:n\}} = Id_{\{j:n\}} = (({}^{\mathrm{t}}T)^{-1})_{\{j:n\}}$. In particular $(y^{-1})_{jj} = (z^{-1})_{jj}$. The proof of (ii) is similar. □

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