Telescopic approach to a formula of ${}_2F_1$-series by Gosper and Ebisu

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Abstract: By means of the telescoping method, we prove an infinite series identity with four free parameters. Its limiting case is utilized, with the help of the Pfaff transformation, not only to present a new proof for a ${}_2F_1$-series identity conjectured by Gosper (1977) and proved recently by Ebisu (2013), but also to establish an extension of the binomial series.

Key words: Classical hypergeometric series; binomial series; telescoping method; Pfaff transformation.

1. Introduction and motivation. According to Bailey [1], the generalized hypergeometric series reads as

$$ {}_{1+p}F_p\left[\begin{matrix} a_0, & a_1, & \cdots, & a_p \\ & b_1, & \cdots, & b_p \end{matrix}\,\middle|\, z\right] = \sum_{k=0}^{\infty} \frac{(a_0)_k (a_1)_k \cdots (a_p)_k}{k!(b_1)_k \cdots (b_p)_k} z^k $$

where the shifted factorial is defined by $(\lambda)_0 = 1$ and

$$ (\lambda)_n = \lambda(\lambda+1)\cdots(\lambda+n-1) \quad \text{for} \quad n = 1, 2, \cdots. $$

In 1977, Gosper [5] conjectured many strange evaluations of hypergeometric series. Gessel and Stanton [4] confirmed most of them subsequently. However, the following identity conjectured by Gosper [5, Eq. a/b1]

$$ (1) \quad {}_2F_1\left[\begin{matrix} 1-a, & b \\ & b+2 \end{matrix}\,\middle|\, \frac{b}{a+b}\right] = (b+1)\left(\frac{a}{a+b}\right)^a $$

was proved only recently by Ebisu [3]. Although this identity did not appear explicitly in Gessel and Stanton's paper [4], they did obtain a formula [4, Eq. 1.9] (see Chu [2, Eq. 3.5e] also)

$$ (2) \quad {}_3F_2\left[\begin{matrix} -n, & b\lambda-\lambda+1, & b-1 \\ & b\lambda+n\lambda-n, & b+1 \end{matrix}\,\middle|\, 1\right] = \frac{b+bn}{b+n}\,\frac{(1-\lambda-n\lambda)_n}{(1-b\lambda-n\lambda)_n} $$

from which there is substantially only one step left towards a completion of proving Gosper's conjecture.

In fact, by letting $\lambda \to \infty$ in the above equality, we get

$$ {}_2F_1\left[\begin{matrix} -n, & b-1 \\ & b+1 \end{matrix}\,\middle|\, \frac{b-1}{b+n}\right] = \frac{b+bn}{b+n}\left(\frac{n+1}{n+b}\right)^n. $$

Replacing b by $b+1$ and n by $n-1$ further, we derive the finite form of (1)

$$ (3) \quad {}_2F_1\left[\begin{matrix} 1-n, & b \\ & b+2 \end{matrix}\,\middle|\, \frac{b}{b+n}\right] = (b+1)\left(\frac{n}{b+n}\right)^n. $$

Finally, Gosper's nonterminating series identity (1) follows in view of the Carlson theorem (cf. Bailey [1, §5.3]).

Observing that (1) resembles very much

$$ {}_1F_0\left[\begin{matrix} 1-a \\ - \end{matrix}\,\middle|\, \frac{b}{a+b}\right] = \left(\frac{a}{a+b}\right)^{a-1} $$

there must be a close connection between (1) and the classical binomial series. The aim of this short article is to offer another proof of (1) and an extension of binomial series by means of telescoping method. This proof turns out much easier than what I expected even though the identity (1) remained unproven for more than 30 years until Ebisu's recent proof [3].

2. Main results and proofs. For the sequence T_k defined by

$$ (4) \qquad T_k = \frac{(a)_k (b)_k}{(c)_k (d)_k} \quad \text{for} \quad k = 0, 1, 2, \cdots $$

it is trivial to check the difference

$$ (5) \qquad \Delta T_k = T_{k+1} - T_k = \frac{(a)_k (b)_k}{(c+1)_k (d+1)_k} $$

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$$\times\ \frac{ab-cd+k(a+b-c-d)}{cd}$$

and the limiting relation

$$T_\infty = \lim_{n\to\infty} T_n = 0 \quad \text{when} \quad \Re(c+d-a-b)>0$$

where we have utilized Euler's asymptotic relation (cf. Stromberg [7, Theorem 7.62])

$$(6)\qquad \Gamma(\lambda+n)\sim(n-1)!n^\lambda \quad \text{as} \quad n\to\infty.$$

By means of telescoping, we get immediately the infinite series identity

$$\sum_{k\ge0}\frac{(a)_k(b)_k}{(c+1)_k(d+1)_k}\frac{ab-cd+k(a+b-c-d)}{ab-cd} = \frac{(T_\infty-T_0)cd}{ab-cd}$$

which can be restated as the following lemma.

Lemma 1. *For the four complex parameters a,b,c,d subject to $\Re(c+d-a-b)>0$, the following summation formula holds*:

$$ {}_4F_3\left[\begin{matrix}1, & a, & b, & 1+\frac{ab-cd}{a+b-c-d}\\ & c+1, & d+1, & \frac{ab-cd}{a+b-c-d}\end{matrix}\ \middle|\ 1\right] = \frac{cd}{cd-ab}.$$

Specifying the parameters in this lemma by $a\to a-b\lambda$, $b\to -n$, $c\to a+n\lambda$ and $d\to b$, we recover the terminating sum of Gessel and Stanton [4, Eq. 5.16]

$$(7)\qquad {}_4F_3\left[\begin{matrix}1, 1+\frac{a}{1+\lambda}, a-b\lambda, -n\\ \frac{a}{1+\lambda}, 1+a+n\lambda, b+1\end{matrix}\ \middle|\ 1\right] = \frac{b(a+n\lambda)}{a(b+n)}$$

which has been rederived by the author [2, Eq. 3.5c] through partial fractions and Gould–Hsu inverse series relations [6].

By making the replacements $b\to Mx$ and $d\to M$, then letting $M\to\infty$ for the equality displayed in Lemma 1, we state the limiting result as follows:

Proposition 2. *Let x be a complex variable with $|x|<1$. The following summation formula holds*:

$$ {}_3F_2\left[\begin{matrix}1, & a, & 1+\frac{ax-c}{x-1}\\ & 1+c, & \frac{ax-c}{x-1}\end{matrix}\ \middle|\ x\right] = \frac{c}{c-ax}.$$

The last equality can be further reduced, by letting $\frac{ax-c}{x-1}=1$ or equivalently $c=1+ax-x$, to the following one (with $|x|<1$)

$$(8)\qquad {}_2F_1\left[\begin{matrix}a, & 2\\ 2+ax-x\end{matrix}\ \middle|\ x\right] = \frac{1+ax-x}{1-x}.$$

Applying the Pfaff transformation (cf. Bailey [1, §2.4])

$$(9)\qquad {}_2F_1\left[\begin{matrix}a, & b\\ & c\end{matrix}\ \middle|\ x\right] = (1-x)^{-a}{}_2F_1\left[\begin{matrix}a, & c-b\\ & c\end{matrix}\ \middle|\ \frac{x}{x-1}\right]$$

to the last series and then performing the replacement $a\to 1+a$ in the resulting equation, we find the following identity (with $|x|<1$)

$$(10)\qquad {}_2F_1\left[\begin{matrix}1+a, & ax\\ & 2+ax\end{matrix}\ \middle|\ \frac{x}{x-1}\right] = (1-x)^a(1+ax).$$

This is equivalent to Gosper's formula (1) under the replacements $a\to -a$ and $x\to -b/a$.

For the series displayed in Proposition 2, we can directly carry on this procedure without losing one parameter. Splitting the series into two ${}_2F_1$-series

$$ {}_3F_2\left[\begin{matrix}1, & a, & 1+\frac{ax-c}{x-1}\\ & 1+c, & \frac{ax-c}{x-1}\end{matrix}\ \middle|\ x\right] = {}_2F_1\left[\begin{matrix}1, a\\ 1+c\end{matrix}\ \middle|\ x\right] + {}_2F_1\left[\begin{matrix}2, & 1+a\\ & 2+c\end{matrix}\ \middle|\ x\right]\frac{ax(x-1)}{(ax-c)(c+1)}$$

and then applying (9) to each ${}_2F_1$-series above, we can unify the resulting expressions together as follows:

$$ {}_3F_2\left[\begin{matrix}a, & c-1, & 1+\frac{(c-1)(ax-c)}{x-1+ax-c}\\ & c+1, & \frac{(c-1)(ax-c)}{x-1+ax-c}\end{matrix}\ \middle|\ \frac{x}{x-1}\right] = (1-x)^a\frac{c}{c-ax}.$$

Writing this equality in variable $y=\frac{x}{x-1}$, we get the following hypergeometric series identity with three free parameters.

Theorem 3. *Let y be a complex variable with $|y|<1$. We have*

$$ {}_3F_2\left[\begin{matrix}a, & c-1, & 1+\frac{(c-1)(c+ay-cy)}{1+c+ay-cy}\\ & c+1, & \frac{(c-1)(c+ay-cy)}{1+c+ay-cy}\end{matrix}\ \middle|\ y\right] = (1-y)^{-a}\frac{c-cy}{c+ay-cy}.$$

What is remarkable is that this identity generalizes, indeed, the well–known binomial series because when $c \to \infty$, it becomes

$$ {}_1F_0\left[\begin{matrix} a \\ - \end{matrix}\middle| y\right] = (1-y)^{-a}. $$

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