Quasi-symmetries and rigidity for determinantal point processes associated with de Branges spaces

Dedicated to Professor Yoichiro Takahashi on the occasion of his 70th birthday

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Abstract: In this note, we show that determinantal point processes on the real line corresponding to de Branges spaces of entire functions are rigid in the sense of Ghosh-Peres and, under certain additional assumptions, quasi-invariant under the group of diffeomorphisms of the line with compact support.

Key words: Quasi-symmetries; rigidity; determinantal point process (DPP); de Branges space.

1. De Branges spaces. Recall that a de Branges function is an entire function E satisfying

$$|E(z)| > |E^{\#}(z)| \quad \text{for } z \in \mathbf{C}_+,$$

where $E^{\#}(z) = \overline{E(\bar{z})}$. We note that such an entire function E does not have zeros in $\mathbf{C}_+$. The de Branges space associated with E is a Hilbert space $B(E)$ of entire functions such that (i) $f|_{\mathbf{R}} \in L^2(\mathbf{R}, |E(\lambda)|^{-2}d\lambda)$, and (ii) $|\frac{f(z)}{E(z)}|, |\frac{f^{\#}(z)}{E(z)}| \le C_f(\operatorname{Im} z)^{-1/2}$ for $z \in \mathbf{C}_+$, where $f|_{\mathbf{R}}$ is the restriction of f on $\mathbf{R}$. Under the condition (i), the condition (ii) is equivalent to the condition that f/E and $f^{\#}/E$ belong to the Hardy space H_2 on the upper-half plane $\mathbf{C}_+$. The de Branges space is a natural generalization of the Paley-Wiener space which is associated with the de Branges function $E(z) = e^{-iaz}$.

The Hilbert space $B(E)$ admits the following reproducing kernel:

$$\Pi(E)(z,w) = \frac{E(z)\overline{E(w)} - E^{\#}(z)\overline{E^{\#}(w)}}{-2\pi i(z-\bar{w})},$$

i.e., for any $f \in B(E)$,

$$f(z) = \int_{\mathbf{R}} \Pi(E)(z,\lambda) f(\lambda) |E(\lambda)|^{-2} d\lambda.$$

The diagonal value is given by

$$\Pi(E)(z,z) = \frac{|E(z)|^2 - |E^{\#}(z)|^2}{4\pi \operatorname{Im} z} > 0 \quad (z \in \mathbf{C} \setminus \mathbf{R}),$$

and

$$\Pi(E)(x,x) = \frac{1}{2\pi}\frac{\partial}{\partial y}|E(x+iy)|^2\Big|_{y=0} \quad (x \in \mathbf{R}).$$

The Hilbert space $B(E)$ is naturally identified with a subspace of $L_2(\mathbf{R}, |E(\lambda)|^{-2}d\lambda)$.

It will, however, be more convenient for us to consider the space

$$\widetilde{B}(E) = \left\{\frac{F(\lambda)}{E(\lambda)}, F \in B(E)\right\},$$

which is then naturally identified with a subspace of $L_2(\mathbf{R})$. Let $\widetilde{\Pi}(E) : L_2(\mathbf{R}) \to \widetilde{B}(E)$ be the corresponding operator of orthogonal projection with kernel

$$\widetilde{\Pi}(E)(z,w) = \Pi(E)(z,w)(E(z)\overline{E(w)})^{-1}.$$

In this note we study determinantal point process (DPP) $\mathbf{P}_{\widetilde{\Pi}(E)}$ on $\mathbf{R}$ corresponding to the locally trace class projection operator $\widetilde{\Pi}(E)$. We recall the necessary definitions.

2. Determinantal point processes.

2.1. Locally trace class operators and their kernels. Let μ be a σ-finite Borel measure on a Polish space S.

Let $\mathscr{I}_1(S,\mu)$ be the ideal of trace class operators $\widetilde{K}: L_2(S,\mu) \to L_2(S,\mu)$ (see e.g. [15] for the

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precise definition); the symbol $\|\widetilde{K}\|_{\mathscr{I}_1}$ will stand for the $\mathscr{I}_1$-norm of the operator $\widetilde{K}$.

Let $\mathscr{I}_{1,\mathrm{loc}}(S,\mu)$ be the space of operators $K\colon L_2(S,\mu)\to L_2(S,\mu)$ such that for any bounded Borel subset $B\subset S$ we have

$$\chi_B K\chi_B \in \mathscr{I}_1(S,\mu).$$

Such an operator K is called a locally trace class operator. Again, we endow the space $\mathscr{I}_{1,\mathrm{loc}}(S,\mu)$ with a countable family of semi-norms

$$\|\chi_B K\chi_B\|_{\mathscr{I}_1} \tag{1}$$

where, as before, B runs through an exhausting family B_n of bounded sets. A locally trace class operator K admits a *kernel*, for which, slightly abusing notation, we use the same symbol K.

2.2. Determinantal point processes. A Borel probability measure $\mathbf{P}$ on $\mathrm{Conf}(S)$, the space of locally finite configurations, is called *determinantal* if there exists an operator $K\in\mathscr{I}_{1,\mathrm{loc}}(S,\mu)$ such that for any bounded measurable function g, for which $g-1$ is supported in a bounded set B, we have

$$\mathbf{E}_{\mathbf{P}}\Psi_g = \det(1+(g-1)K\chi_B), \tag{2}$$

where $\Psi_g(X)=\prod_{x\in X} g(x)$ for $X\in\mathrm{Conf}(S)$. The Fredholm determinant in (2) is well-defined since $K\in\mathscr{I}_{1,\mathrm{loc}}(E,\mu)$. The equation (2) determines the measure $\mathbf{P}$ uniquely.

For any pairwise disjoint bounded Borel sets $B_1,\dots,B_l\subset S$ and any $z_1,\dots,z_l\in\mathbf{C}$ from (2) we have

$$\mathbf{E}_{\mathbf{P}} z_1^{\#_{B_1}}\cdots z_l^{\#_{B_l}} = \det\left(1+\sum_{j=1}^{l}(z_j-1)\chi_{B_j}K\chi_{\sqcup_i B_i}\right).$$

If K belongs to $\mathscr{I}_{1,\mathrm{loc}}(S,\mu)$, then, throughout the paper, we denote the corresponding determinantal measure by $\mathbf{P}_K$. If $K\in\mathscr{I}_{1,\mathrm{loc}}(S,\mu)$, then the existence of the probability measure $\mathbf{P}_K$ is guaranteed ([16,19]).

For further results and background on determinantal point processes, see e.g. [8,10–12,17–19].

3. The integrable form of the reproducing kernel. Our aim in this note is to study rigidity (in the sense of Ghosh and Peres) and the quasi-symmetries of the point process $\mathbf{P}_{\widetilde{\Pi}(E)}$. We start by fixing some notation. For a de Branges function E, we set

$$A(z)=\frac{E(z)+E^{\#}(z)}{2},\ B(z)=\frac{E(z)-E^{\#}(z)}{2i}.$$

The kernel of the operator $\widetilde{\Pi}(E)$, essentially the reproducing kernel of our de Branges space, takes the form

$$\widetilde{\Pi}(E)(x,y)=\frac{1}{\pi}\frac{A(x)B(y)-B(x)A(y)}{(x-y)E(x)\overline{E(y)}},\ x,y\in\mathbf{R}.$$

Slightly abusing notation, we keep the same symbol for the kernel as well as for the operator. For the diagonal values, it is easy to see that

$$\begin{aligned}\widetilde{\Pi}(E)(x,x) &= \frac{1}{2\pi}|E(x)|^{-2}\frac{\partial}{\partial y}|E(x+iy)|^2\Big|_{y=0}\\ &= \frac{1}{\pi}\frac{\partial}{\partial y}\log|E(x+iy)|\Big|_{y=0}.\end{aligned} \tag{3}$$

The kernel $\widetilde{\Pi}(E)$ has an *integrable* form. Corollary 2.2 in [2] now implies the rigidity, in the sense of Ghosh and Peres [8,9], of the determinantal measure $\mathbf{P}_{\widetilde{\Pi}(E)}$. Before giving the notion of rigidity and our results, we provide some examples of determinantal point processes (DPPs).

4. Examples of determinantal point processes associated with de Branges spaces. Here we give some examples of DPPs associated with de Branges spaces.

Example 1 (A class of orthogonal polynomial ensembles)**.** Let $E(z)=\prod_{i=1}^{n}(z+a_i)$ for $a_i\in\mathbf{C}_+$. In this case, $B(E)$ is the space of polynomials of degree less than or equal to $n-1$. The corresponding DPP is the n-th orthogonal polynomial ensemble with weight $|E(\lambda)|^{-2}$. In particular, its intensity is given by

$$\widetilde{\Pi}(E)(x,x)=\frac{1}{\pi}\sum_{i=1}^{n}\frac{\operatorname{Im}a_i}{|x+a_i|^2}.$$

Example 2 (Sine-process)**.** The Paley-Wiener space, for which $E(z)=e^{-iaz}$ $(a>0)$, $A(z)=\cos az$, $B(z)=-\sin az$ yields the sine-kernel $\widetilde{\Pi}(E)(x,y)=\frac{\sin a(x-y)}{\pi(x-y)}$.

Example 3 (Eigenfunction expansion for Schrödinger equation)**.** Fix $\ell\in(0,\infty]$. For $V\in L^1_{\mathrm{loc}}([0,\ell))$, we consider the Schrödinger equation

$$-\varphi_\lambda''+V\varphi_\lambda=\lambda\varphi_\lambda\quad(\lambda\in\mathbf{C})$$

with $\varphi_\lambda(0)=1$ and $\varphi_\lambda'(0)=0$. The solution $\varphi_\lambda(x)$ is jointly continuous in (λ,x) and entire in λ. Suppose that the right boundary $x=\ell$ is of the limit circle type. Then, for each fixed $b \in (0, \ell)$,

$$E_b(z) = \varphi_z(b) + i\varphi_z'(b)$$

defines a de Branges function. In this case,

$$\begin{aligned}\Pi(E_b)(z,w) &= \frac{1}{\pi}\frac{\varphi_z(b)\overline{\varphi_w'(b)} - \varphi_z'(b)\overline{\varphi_w(b)}}{z-\bar{w}} \\ &= \frac{1}{\pi}\int_0^b \varphi_z(t)\overline{\varphi_w(t)}dt.\end{aligned}$$

The intensity of the corresponding DPP is given by

$$\widetilde{\Pi}(E_b)(\lambda,\lambda) = \frac{1}{\pi}\frac{\displaystyle\int_0^b |\varphi_\lambda(t)|^2 dt}{|\varphi_\lambda(b)|^2 + |\varphi_\lambda'(b)|^2}.$$

5. Ghosh-Peres rigidity. Given a bounded subset $B \subset \mathbf{R}$ and a configuration $X \in \mathrm{Conf}(\mathbf{R})$, let $\#_B(X)$ stand for the number of particles of X lying in B. Given a Borel subset $C \subset \mathbf{R}$, we let $\mathcal{F}_C$ be the σ-algebra generated by all random variables of the form $\#_B, B \subset C$. If $\mathbf{P}$ is a point process on $\mathbf{R}$ then we write $\mathcal{F}_C^{\mathbf{P}}$ for the $\mathbf{P}$-completion of $\mathcal{F}_C$.

Definition (Ghosh and Peres [8,9])**.** A point process $\mathbf{P}$ is called **rigid** if for any bounded Borel subset B the random variable $\#_B$ is $\mathcal{F}_{\mathbf{R}\setminus B}^{\mathbf{P}}$-measurable.

Theorem 1. *The determinantal measure $\mathbf{P}_{\widetilde{\Pi}(E)}$ is rigid in the sense of Ghosh and Peres.*

Proof. By Corollary 2.2 in [2], we need to establish the existence of $R > 0$, $C > 0$ and $\varepsilon > 0$ such that for all $|x| < R$ we have $|A(x)| \le C|x|^{-1/2+\varepsilon}|E(x)|$; $|B(x)| \le C|x|^{-1/2+\varepsilon}|E(x)|$ and for all $|x| > R$ we have $|A(x)| \le C|x|^{1/2-\varepsilon}|E(x)|$; $|B(x)| \le C|x|^{1/2-\varepsilon}|E(x)|$; and these conditions hold since $|A(x)|, |B(x)| \le |E(x)|$. □

Proposition 8.1 in [4] now implies the following

Corollary 2. *For any $k, l \in \mathbf{N}$, $k \neq l$, for almost any k-tuple $(p_1, \ldots, p_k)$ and almost any l-tuple $(q_1, \ldots, q_l)$ of distinct points in $\mathbf{R}$, the reduced Palm measures $\mathbf{P}^{p_1,\ldots,p_k}_{\widetilde{\Pi}(E)}$ and $\mathbf{P}^{q_1,\ldots,q_l}_{\widetilde{\Pi}(E)}$ are mutually singular.*

6. Quasi-symmetries. We next give sufficient conditions for the equivalence of Palm measures of the same order. Let $p_1, \ldots, p_l, q_1, \ldots, q_l \in \mathbf{R}$ be distinct. For $R > 0$, $\varepsilon > 0$ and a configuration X on $\mathbf{R}$, similarly to [1], we introduce an approximation of the Radon-Nikodym density $d\mathbf{P}^{p_1,\ldots,p_l}_{\widetilde{\Pi}(E)}/d\mathbf{P}^{q_1,\ldots,q_l}_{\widetilde{\Pi}(E)}$ as the real-valued normalized multiplicative functional

$$\begin{aligned}&\overline{\Psi}_{R,\varepsilon}(p_1, \ldots, p_l; q_1, \ldots, q_l; X) \\ &\quad = C(R,\varepsilon) \times \prod_{x\in X, |x|\le R, \min|x-q_i|\ge\varepsilon} \prod_{i=1}^{l}\left(\frac{x-p_i}{x-q_i}\right)^2,\end{aligned}$$

where the constant $C(R,\varepsilon)$ is chosen in such a way that

$$(4) \quad \int_{\mathrm{Conf}(\mathbf{R})} \overline{\Psi}_{R,\varepsilon}(p_1,\ldots,p_l; q_1,\ldots,q_l; X)d\mathbf{P}^{q_1,\ldots,q_l}_{\widetilde{\Pi}(E)} = 1.$$

We will often need the following assumption on our de Branges function E:

$$(5) \quad \int_{\mathbf{R}} \frac{\frac{\partial}{\partial y}|E(x+iy)|^2|_{y=0}}{(1+x^2)|E(x)|^2}dx < +\infty.$$

Given our de Branges function E, there exists a nondecreasing continuous function ϕ on $\mathbf{R}$ such that $E(x)\exp(i\phi(x))$ is real for all $x \in \mathbf{R}$. The function $\phi(x)$ is called a *phase function* associated with $E(z)$. We note that

$$(6) \quad \phi'(x) = \pi\widetilde{\Pi}(E)(x,x) > 0 \quad (\forall x \in \mathbf{R}).$$

(See de Branges [5] Problem 48.) From (3) and (6), the assumption (5) can equivalently be reformulated as follows:

$$(7) \quad \int_{\mathbf{R}} \frac{\phi'(x)}{1+x^2}dx = \int_{\mathbf{R}} \frac{d\phi(x)}{1+x^2} < \infty.$$

It is known that there exists a $p > 0$ such that

$$\begin{aligned}(8) \quad &\frac{\partial}{\partial y}\log|E(x+iy)| \\ &\quad = py + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{(t-x)^2+y^2}d\phi(t)\end{aligned}$$

if E has no real zeros and $|E(x+iy)|$ is a nondecreasing function of $y > 0$ for each $x \in \mathbf{R}$. (See de Branges [5] Problem 63.)

Remark. If E is of exponential type and has no real zeros, then the condition (7) holds. Indeed, if E is of exponential type, then $|E(x+iy)|$ is nondecreasing in $y > 0$ (see Dym [6] Lemma 4.1). Setting $x = 0$ and $y = 1$ in (8) yields (7). In particular, if E is *short* in the sense that $B(E)$ is closed under the map $f(z) \mapsto \frac{f(z)-f(i)}{z-i}$ (see Dym and McKean [7] Proposition 6.2.2), then (7) holds.

Proposition 3. *Let E be a de Branges function satisfying* (7)*. Then the limit*

$$\begin{aligned}&\overline{\Psi}(p_1,\ldots,p_l; q_1,\ldots,q_l; X) \\ &\quad = \lim_{R\to\infty,\varepsilon\to 0}\overline{\Psi}_{R,\varepsilon}(p_1,\ldots,p_l; q_1,\ldots,q_l; X)\end{aligned}$$

exists in $L_1\Big(\mathrm{Conf}(\mathbf{R}), \mathbf{P}^{q_1,\ldots,q_l}_{\widetilde{\Pi}(E)}\Big)$, almost surely along *a subsequence, and satisfies*

$$(9) \qquad \int_{\mathrm{Conf}(\mathbf{R})} \overline{\Psi}(p_1,\dots,p_l;q_1,\dots,q_l;X)d\mathbf{P}^{q_1,\dots,q_l}_{\widetilde{\Pi}(E)} = 1.$$

Corollary 4.12 in [1] now directly implies

Proposition 4. *Let E be a de Branges function satisfying* (7). *Then for any distinct points $p_1,\dots,p_l,q_1,\dots,q_l \in \mathbf{R}$, the corresponding reduced Palm measures are equivalent, and we have*

$$\frac{d\mathbf{P}^{p_1,\dots,p_l}_{\widetilde{\Pi}(E)}}{d\mathbf{P}^{q_1,\dots,q_l}_{\widetilde{\Pi}(E)}}(X) = \overline{\Psi}(p_1,\dots,p_l;q_1,\dots,q_l;X).$$

Remark. Similar results to Corollary 2 and Proposition 4 for the Ginibre point process were obtained in [14] and for generalized Ginibre point processes in [4].

Theorem 1.5 in [1] directly implies the following

Proposition 5. *Let E be a de Branges function satisfying* (7). *Let $F : \mathbf{R} \to \mathbf{R}$ be a diffeomorphism acting as the identity beyond a bounded open set $V \subset \mathbf{R}$. For $\mathbf{P}_{\widetilde{\Pi}(E)}$-almost every configuration $X \in \mathrm{Conf}(\mathbf{R})$ the following holds. If $X \bigcap V = \{q_1,\dots,q_l\}$, then*

$$(10) \qquad \frac{d\mathbf{P}_{\widetilde{\Pi}(E)} \circ F}{d\mathbf{P}_{\widetilde{\Pi}(E)}}(X) = \overline{\Psi}(F(q_1),\dots,F(q_l);q_1,\dots,q_l;X) \times \frac{\det(\widetilde{\Pi}(E)(F(q_i),F(q_j)))_{i,j=1,\dots,l}}{\det(\widetilde{\Pi}(E)(q_i,q_j))_{i,j=1,\dots,l}} \times F'(q_1)\cdots F'(q_l).$$

Remark. The open set V can be chosen in many ways; the resulting value of the Radon-Nikodym derivative is of course the same.

Remark. As in [1], F can, more generally, be a compactly supported Borel automorphism preserving the Lebesgue measure class. In this case, the derivative F' in (10) should be replaced by the Radon-Nikodym derivative of the Lebesgue measure under F. In the discrete setting, similar results were obtained in [13] in the case of the Gamma-kernel and in [1] in the generality of integrable kernels.

Remark. Conditional measures of our DPPs can now also be found using the results of [3].

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