Radial symmetry and its breaking in the Caffarelli-Kohn-Nirenberg type inequalities for $p = 1$

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Abstract: The main purpose of this article is to study the Caffarelli-Kohn-Nirenberg type inequalities (1.2) with $p = 1$. We show that symmetry breaking of the best constants occurs provided that a parameter $|\gamma|$ is large enough. In the argument we effectively employ equivalence between the Caffarelli-Kohn-Nirenberg type inequalities with $p = 1$ and the isoperimetric inequalities with weights.

Key words: CKN-type inequality; symmetry break; weighted Hardy-Sobolev inequality; best constant.

1. Introduction. The Caffarelli-Kohn-Nirenberg type inequalities (the CKN-type inequalities) were introduced in [4] originally as rather general multiplicative inequalities with weights being powers of distance from the origin, and they have been studied eagerly afterwards by many authors (see e.g. [1,2,5,6,8,10,12,17–20,23,24,26]). Symmetry breaking of extremal functions for the CKN-type inequalities is also studied intensively in the case where $p = 2$ (see e.g. [11,13–16] and see also [3,21] for $p > 1$). Recently in [22] the second author systematically investigated the CKN-type inequalities involving critical and supercritical weights. More precisely, validity of inequalities, existence of extremal functions, continuity of best constants, symmetry of extremal functions and symmetry breaking of the best constants were established. For this purpose the condition $p > 1$ was assumed in [22] in most cases, but some results on symmetry were studied including the case where $p = 1$ as well. On a basis of these observation, we study in the present paper symmetry property of the best constant and its breaking phenomena for the CKN-type inequalities when $p = 1$. We show that symmetry breaking occurs provided that a parameter $|\gamma|$ is large enough. In the argument we effectively employ equivalence between the CKN-type inequalities with $p = 1$ and the isoperimetric inequalities with weights. The full proofs will be given in the paper [7].

First we define a class of weighted function. For $\alpha \in \mathbf{R}$ and $n \geq 1$ we set

$$I_\alpha(x) = I_\alpha(|x|) = |x|^{\alpha - n} \quad \text{for } x \in \mathbf{R}^n \backslash \{0\}. \tag{1.1}$$

When $0 < \alpha < n$ holds, I_α is called a Riesz kernel of order α. Then the classical CKN-type inequalities with $p \geq 1$ are represented in the following way: For $\gamma \in \mathbf{R}\backslash\{0\}$, there exists a positive number C such that we have

$$\left(\int_{\mathbf{R}^n} |\nabla u(x)|^p I_{p(1+\gamma)}(x)dx\right)^{1/p} \geq C\left(\int_{\mathbf{R}^n} |u(x)|^q I_{q\gamma}(x)dx\right)^{1/q} \tag{1.2}$$

for any $u \in C_c^\infty(\mathbf{R}^n\backslash\{0\})$, where $1 \leq p \leq q < \infty$, $1/p - 1/q \leq 1/n$, and C depends only on p, q, γ and n. As was mentioned before, when $\gamma > 0$ holds, the CKN-type inequalities were introduced in [4] as a part of multiplicative interpolation inequalities, and later in [22] the CKN-type inequalities were further investigated for all $\gamma \in \mathbf{R}$. In the present paper, we shall give a simple proof of the CKN-type inequalities (1.2) for $p = 1$ with the best constant $S^{1,q;\gamma}$ defined below, using equivalence between the CKN-type inequalities with $p = 1$ and the isoperimetric inequalities with weights. We note that in [19] a class of weighted Sobolev inequalities were studied by using isoperimetric inequalities with general weight functions. To give a precise definition of the best constant $S^{1,q;\gamma}$ it is convenient to adopt the following function spaces and relating norms including the case where $p \geq 1$:

Definition 1.1. Let $1 \leq p \leq q < \infty$, $\gamma \in$

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$\mathbf{R}\backslash\{0\}$ and let $u : \mathbf{R}^n \to \mathbf{R}$.

1. $\|u\|_{L^q_\gamma(\mathbf{R}^n)} = \left(\int_{\mathbf{R}^n} |u|^q I_{q\gamma}(x)\, dx\right)^{1/q}$,
   $\|\nabla u\|_{L^p_{1+\gamma}(\mathbf{R}^n)} = \left(\int_{\mathbf{R}^n} |\nabla u|^p I_{p(1+\gamma)}(x)\, dx\right)^{1/p}$.
2. $L^q_\gamma(\mathbf{R}^n) = \{u : \mathbf{R}^n \to \mathbf{R} \mid \|u\|_{L^q_\gamma(\Omega)} < \infty\}$.
3. By $W^{1,p}_{\gamma,0}(\mathbf{R}^n)$ we denote the completion of $C^\infty_{\mathrm{c}}(\mathbf{R}^n\backslash\{0\})$ w.r.t. the norm $u \mapsto \|\nabla u\|_{L^p_{1+\gamma}(\mathbf{R}^n)}$.
4. For any function space $V(\mathbf{R}^n)$ on $\mathbf{R}^n$, we set $V(\mathbf{R}^n)_{\mathrm{rad}} = \{u \in V(\mathbf{R}^n) \mid u \text{ is radial}\}$.

Here we remark the following fundamental properties concerning with the density of smooth functions.

Proposition 1.1. *Assume that* $1 \le p < \infty$ *and* $\gamma \in \mathbf{R}$.

1. *If* $\gamma > 0$, *then* $C^\infty_{\mathrm{c}}(\mathbf{R}^n) \subset W^{1,p}_{\gamma,0}(\mathbf{R}^n)$ *and* $C^\infty_{\mathrm{c}}(\mathbf{R}^n)$ *are densely contained in* $W^{1,p}_{\gamma,0}(\mathbf{R}^n)$.
2. *If* $\gamma < 0$, *then* $C^\infty_{\mathrm{c}}(\mathbf{R}^n) \not\subset W^{1,p}_{\gamma,0}(\mathbf{R}^n)$.
3. $C^\infty_{\mathrm{c}}(\mathbf{R}^n\backslash\{0\})$ *and* $C^\infty_{\mathrm{c}}(\mathbf{R}^n\backslash\{0\})_{\mathrm{rad}}$ *are densely contained in* $W^{1,p}_{\gamma,0}(\mathbf{R}^n)$ *and* $W^{1,p}_{\gamma,0}(\mathbf{R}^n)_{\mathrm{rad}}$, *respectively.*

Let us introduce more notations including the case where $p \ge 1$.

Definition 1.2. Let $1 \le p \le q < \infty$, $1/p - 1/q \le 1/n$ and $\gamma \ne 0$.

(1.3) $$E^{p,q;\gamma}[u] = \frac{\|\nabla u\|_{L^p_{1+\gamma}(\mathbf{R}^n)}}{\|u\|_{L^q_\gamma(\mathbf{R}^n)}} \text{ for } u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n)\backslash\{0\}.$$

(1.4) $$S^{p,q;\gamma} = \inf\{E^{p,q;\gamma}[u] \mid u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n)\backslash\{0\}\}.$$

(1.5) $$S^{p,q;\gamma}_{\mathrm{rad}} = \inf\{E^{p,q;\gamma}[u] \mid u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n)_{\mathrm{rad}}\backslash\{0\}\}.$$

Definition 1.3. Let ω_n be a surface area of an n-dimensional unit ball. For $1 \le q < \infty$, we set

(1.6) $$S_{1,q} = \omega_n^{1-1/q} q^{1/q}.$$

First of all we state the CKN-type inequalities with the best constants for all $\gamma \ne 0$.

Theorem 1.1. *Let* $n \ge 1$ *and* $\gamma \in \mathbf{R}\backslash\{0\}$. *Assume that* $1 \le q \le n/(n-1)$ *if* $n > 1$; $1 \le q < \infty$ *if* $n = 1$. *Then we have* $S^{1,q;\gamma}_{\mathrm{rad}} \ge S^{1,q;\gamma} > 0$ *and the following inequalities: For any* $u \in C^\infty_{\mathrm{c}}(\mathbf{R}^n\backslash\{0\})$

(1.7) $$\int_{\mathbf{R}^n} |\nabla u(x)| I_{1+\gamma}(x) dx \ge S^{1,q;\gamma} \left(\int_{\mathbf{R}^n} |u(x)|^q I_{q\gamma}(x) dx\right)^{1/q}.$$

The inequality (1.7) follows directly from Theorem 1.2, and the positivity of $S^{1,q;\gamma}$ follows from Proposition 2.1;1 and Theorem 1.3.

Definition 1.4. An open set $M \subset \mathbf{R}^n$ is called admissible if $\overline{M}$ is bounded and ∂M is a C^∞ manifold.

In the next we state equivalence between the CKN-type inequalities with $p = 1$ and isoperimetric inequalities with weights (the proof is given in Section 3).

Theorem 1.2. *Let* $n \ge 1$ *and* $\gamma \in \mathbf{R}\backslash\{0\}$. *Assume that* $1 \le q \le n/(n-1)$ *if* $n > 1$; $1 \le q < \infty$ *if* $n = 1$. *Assume that* M *is an arbitrary admissible open set of* $\mathbf{R}^n$. *Then the CKN-type inequalities are equivalent to the corresponding weighted isoperimetric inequalities with the same best constants, which are given by the following*

(1.8) $$\int_{\partial M} I_{1+\gamma}(x)\, dS \ge S^{1,q;\gamma} \left(\int_M I_{q\gamma}(x)\, dx\right)^{1/q},$$

where S *denotes the* $(n-1)$*-dimensional Lebesgue measure.*

Remark 1.1. By A_{rad} we denote a family of all admissible open sets of $\mathbf{R}^n$ which are radially symmetric with respect to the origin. From Theorem 1.2 and its proof we easily see that

(1.9) $$S^{1,q;\gamma}_{\mathrm{rad}} = \inf_{M \in A_{\mathrm{rad}}\backslash\{\phi\}} \frac{\int_{\partial M} I_{1+\gamma}(x)\, dS}{\left(\int_M I_{q\gamma}(x)\, dx\right)^{1/q}}.$$

In the next we describe symmetric properties and important relations among $S^{1,q;\gamma}_{\mathrm{rad}}$ and $S^{1,q;\gamma}$.

Theorem 1.3 (Symmetry). *Let* $n \ge 1$. *Assume that* $1 \le q \le n/(n-1)$ *if* $n > 1$; $1 \le q < \infty$ *if* $n = 1$. *Let* $\tau_{1,q} = 1 - 1/q$ $(\le 1/n)$. *Then it holds that:*

1. $S^{1,q;\gamma} = S^{1,q;-\gamma}$, $S^{1,q;\gamma}_{\mathrm{rad}} = S^{1,q;-\gamma}_{\mathrm{rad}}$ *for* $\gamma \ne 0$.
2. $S^{1,q;\gamma}_{\mathrm{rad}} = S_{1,q}|\gamma|^{1-\tau_{1,q}}$ *for* $\gamma \ne 0$.
3. $S^{1,q;\gamma} = S^{1,q;\gamma}_{\mathrm{rad}} = S_{1,q}|\gamma|^{1-\tau_{1,q}}$ *for* $0 < |\gamma| \le n-1$, $n > 1$.

Proof. The assertion 1 of this theorem is already established in Proposition 3.1; [22] and the assertions 2 and 3 are proved by direct calculations using Theorem 1.2 and its remark. □

Now we state main theorems in this article.

Theorem 1.4 (Symmetry breaking). *Assume that* q *is fixed such as* $1 \le q \le n/(n-1)$ *if* $n > 1$; $1 \le q < \infty$ *if* $n = 1$. *Then we have the followings:*

1. *If* $n > 1$, *then we have* $S^{1,q;\gamma}_{\mathrm{rad}} > S^{1,q;\gamma}$ *for sufficiently large* $|\gamma|$.
2. *If* $n = 1$, *then we have* $S^{1,q;\gamma}_{\mathrm{rad}} = 2^{1-1/q} S^{1,q;\gamma}$ $(> S^{1,q;\gamma})$ *for any* $\gamma \in \mathbf{R}\backslash\{0\}$.

Remark 1.2.

1. When $q = 1$ holds, the CKN-type inequality becomes the weighted Hardy inequality and

the best constant $S^{1,1;\gamma}$ coincides with the one restricted in $W^{1,1}_{\gamma,0}(\mathbf{R}^n)_{\mathrm{rad}}$:

$$S^{1,1;\gamma} = S^{1,1;\gamma}_{\mathrm{rad}} = |\gamma| \quad \text{for } \gamma \neq 0. \tag{1.10}$$

2. For $n > 1$, the number; $S^{1,1^*;n-1} = S^{1,1^*;n-1}_{\mathrm{rad}} = n\Big(\dfrac{\omega_n}{n}\Big)^{1/n}$ coincides with the best constant of the Sobolev inequality (c.f. [25]) and $S^{1,1^*;n-1}$ is attained by characteristic functions of balls which do not belong to $W^{1,1}_{n-1,0}(\mathbf{R}^n)$ but to the space of functions of bounded variation. Therefore the space $W^{1,1}_{\gamma,0}(\mathbf{R}^n)$ is too small to discuss existences of extremal functions of the best constants. In the future we shall study existences of extremal functions using suitable spaces of functions of bounded variation (c.f. [9]).

Definition 1.5. For $1 \leq q \leq n/(n-1)$ if $n > 1; 1 \leq q < \infty$ if $n = 1$, we set

$$\Gamma_1(q) = \sup\{\gamma > 0 : S^{1,q;\gamma} = S^{1,q;\gamma}_{\mathrm{rad}}\}. \tag{1.11}$$

It follows from Theorem 1.3; 3 and Theorem 1.4 that we have $n - 1 \leq \Gamma_1(q) < \infty$ $(1 \leq q \leq n/(n-1)$ if $n > 1; 1 \leq q < \infty$ if $n = 1)$. More precisely we have:

Theorem 1.5. *Assume that* $1 \leq q \leq n/(n-1)$ *if* $n > 1; 1 \leq q < \infty$ *if* $n = 1$. *Then we have the followings*:

1. *If* $\gamma \in (0, \Gamma_1(q)]$, *then* $S^{1,q;\gamma} = S^{1,q;\gamma}_{\mathrm{rad}}$.
2. *If* $\gamma \in (\Gamma_1(q), \infty)$, *then* $S^{1,q;\gamma} < S^{1,q;\gamma}_{\mathrm{rad}}$.

Our article is organized in the following way. In Section 2 we introduce useful change of variables. In Section 3 we give sketch of proofs of Theorems 1.2, 1.4 and 1.5.

2. Change of variables. Here we see relations among the best constants by a method of change of variables.

Definition 2.1. For $\beta > 0$, we set $Y_\beta(y) = |y|^{\beta-1}y$ for $y \in \mathbf{R}^n$.

Definition 2.2. Let $\beta > 0$ and $u : \mathbf{R}^n \to \mathbf{R}$. $T_\beta u(y) = u(Y_\beta(y)) = u(|y|^{\beta-1}y)$ for $y \in \mathbf{R}^n$.

By a calculation we have the next lemma.

Lemma 2.1. *Assume that* $1 \leq p \leq q < \infty$, $\gamma > 0$, *and* $\beta > 0$. *Then we have the followings*:

$$\|u\|_{L^q_\gamma(\mathbf{R}^n)} = \beta^{1/q}\|T_\beta u\|_{L^q_{\beta\gamma}(\mathbf{R}^n)} \quad \textit{for } u \in L^q_\gamma(\mathbf{R}^n),$$

$$\|\nabla u\|_{L^p_{1+\gamma}(\mathbf{R}^n)} = \frac{1}{\beta^{1/p'}}\left\|\left(\left|\frac{\partial}{\partial r}[T_\beta u]\right|^2 + \frac{\beta^2}{r^2}|\Lambda[T_\beta u]|^2\right)^{1/2}\right\|_{L^p_{1+\beta\gamma}(\mathbf{R}^n)} \quad \textit{for } u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n).$$

As an application of Lemma 2.1, let us prepare some estimates of the best constants as a basic tool for the proofs of main results.

Proposition 2.1. *Let* $n \geq 1$. *Assume that* $1 \leq q \leq n/(n-1)$ *if* $n > 1; 1 \leq q < \infty$ *if* $n = 1$. *Let* $\tau_{1,q} = 1 - 1/q$ $(\leq 1/n)$. *Then it holds that*:

1. $\left|\dfrac{\gamma}{\overline{\gamma}}\right|^{1-\tau_{1,q}} S^{1,q;\overline{\gamma}} \leq S^{1,q;\gamma} \leq \left|\dfrac{\overline{\gamma}}{\gamma}\right|^{\tau_{1,q}} S^{1,q;\overline{\gamma}}$ *for* $0 < |\gamma| \leq |\overline{\gamma}|$.
2. $\dfrac{1}{2-(n-1)/\gamma} S^{1,1^*;n-1} \leq S^{1,1^*;\gamma} \leq S^{1,1^*;n-1} = S^{1,1^*;n-1}_{\mathrm{rad}}$ *for* $|\gamma| \geq n-1$ *and* $n > 1$, *where* $1^* = n/(n-1)$.

Proof. These estimates are simple variants of the assertions 4 and 5 of Theorem 2.2 in [22]. □

3. Proofs of Theorems. In the proofs of theorems, we may assume that $\gamma > 0$ by virtue of the assertion 1 of Theorem 1.3. Let us prepare two lemmas without proofs (see e.g. Theorem 1.24 and Lemma 1 in Section 1.35 in [24]).

Lemma 3.1 (Coarea formula). *Assume that* Ω *is domain of* $\mathbf{R}^n$ *and* Φ *is a nonnegative Borel measurable function on* Ω. *Then we have for* $u \in C^{0,1}_c(\Omega)$

$$\int_\Omega \Phi(x)|\nabla u(x)|\,dx = \int_0^\infty dt \int_{|u(t)|=t} \Phi(x)\,dS. \tag{3.1}$$

Here $C^{0,1}_c(\Omega)$ *denotes the space of all Lipschiz continuous functions with compact support in* Ω, *and* S *denotes the* $(n-1)$*-dimensional Lebesgue measure.*

Lemma 3.2. *Assume that* f *is a nonnegative non-increasing function on* $(0,\infty)$ *and* $p \geq 1$. *Then we have*

$$\int_0^\infty (f(x))^p\,d(x^p) \leq \left(\int_0^\infty f(x)\,dx\right)^p. \tag{3.2}$$

Proof of Theorem 1.2. Since it is standard to show the implication (1.7) to (1.8), we assume that (1.8). Let us set $M_t = \{|u(x)| > t\}$ for any $u \in C^\infty_c(\mathbf{R}^n\backslash\{0\})$. Since we may assume that M_t is admissible for almost all t, by Lemma 3.1 and Lemma 3.2 we have

$$\begin{aligned}
&\int_{\mathbf{R}^n} |\nabla u(x)| I_{1+\gamma}(x)\,dx \\
&= \int_0^\infty dt \int_{\partial M_t} I_{1+\gamma}(x)\,dS \\
&\geq S^{1,q;\gamma} \int_0^\infty dt \biggl(\int_{M_t} I_{q\gamma}(x)\,dx \biggr)^{1/q} \\
&\geq S^{1,q;\gamma} \biggl(\int_0^\infty d(t^q) \int_{M_t} I_{q\gamma}(x)\,dx \biggr)^{1/q} \\
&= S^{1,q;\gamma} \biggl(\int_{\mathbf{R}^n} |u(x)|^q I_{q\gamma}(x)\,dx \biggr)^{1/q}.
\end{aligned}$$

Hence we have (1.7) with the same best constant $S^{1,q;\gamma}$. □

Proof of Theorem 1.4. First we prove Theorem 1.4 when $n > 1$. For $0 < \delta < 1$ we set $A_\delta = \{1-\delta < |x| < 1\}$ and $B_{2\delta} = \{|x| < 2\delta\}$. By C and C' we denote positive numbers depending only on the dimension of the space. Let $e = (1, 0, \cdots, 0)$ and set $M = D_\delta = A_\delta \cap \{e + B_{2\delta}\}$. We prepare fundamental estimates involving A_δ and D_δ. First we see that there is a $\delta_0 > 0$ and a positive number C such that we have

$$\operatorname{vol}(D_\delta)/\operatorname{vol}(A_\delta) \geq C\delta^{n-1} \quad (0 < \delta < \delta_0).$$

Then by a direct calculation we can prove

$$\int_{\partial M} I_{1+\gamma}(x)\,dS \leq C\delta^{n-1} \int_{\partial B_1} I_{1+\gamma}(x)\,dS \qquad (0 < \delta < \delta_0). \tag{3.3}$$

On the other hand, we see

$$\begin{aligned}
\int_{A_\delta} I_{q\gamma}(x)\,dx &= \int_{\partial B_1} dS \int_{1-\delta}^1 r^{q\gamma-n} r^{n-1}\,dr \\
&= (1-(1-\delta)^{q\gamma}) \int_{B_1} I_{q\gamma}(x)\,dx.
\end{aligned}$$

Let $x = (x_1, x')$ and $D'_\delta = A_\delta \cap \{|x'| < \delta x_1\}$. Then $D'_\delta \subset D_\delta = M$ and $\operatorname{vol}(D'_\delta)/\operatorname{vol}(D_\delta) \geq C$ hold. Thus

$$\begin{aligned}
\int_M I_{q\gamma}(x)\,dx &\geq \int_{D'_\delta} I_{q\gamma}(x)\,dx \\
&= \frac{\operatorname{vol}(D'_\delta)}{\operatorname{vol}(A_\delta)} \int_{A_\delta} I_{q\gamma}(x)\,dx \\
&\geq C\delta^{n-1} \int_{A_\delta} I_{q\gamma}(x)\,dx.
\end{aligned} \tag{3.4}$$

It follows from (3.3) and (3.4) that we have

$$\begin{aligned}
&\frac{\int_{\partial M} I_{1+\gamma}(x)\,dS}{(\int_M I_{q\gamma}(x)\,dx)^{1/q}} \\
&\leq \frac{C\delta^{n-1} \int_{\partial B_1} I_{1+\gamma}(x)\,dS}{C'\delta^{(n-1)/q}(1-(1-\delta)^{q\gamma})^{1/q} (\int_{B_1} I_{q\gamma}(x)\,dx)^{1/q}} \\
&\leq C\delta^{(n-1)(1-1/q)} \frac{\int_{\partial B_1} I_{1+\gamma}(x)\,dS}{(\int_{B_1} I_{q\gamma}(x)\,dx)^{1/q}} \\
&\qquad \text{(for a sufficiently large } \gamma) \\
&= C\delta^{(n-1)(1-1/q)} S^{1,q;\gamma}_{\mathrm{rad}}.
\end{aligned} \tag{3.5}$$

Hence for sufficiently small δ, we have $C\delta^{(n-1)(1-1/q)} < 1$ (see Remark 3.1).

We proceed to the case where $n = 1$. By S we denote the counting measure. We have for $a > 0$ and $\gamma > 0$

$$S^{1,q;\gamma}_{\mathrm{rad}} = \inf_a \frac{\int_{-a}^a |x|^\gamma\,dS}{(\int_{-a}^a |x|^{q\gamma-1}\,dx)^{1/q}} = 2^{1-1/q}(q\gamma)^{1/q}. \tag{3.6}$$

On the other hand we have for $a < 0 < b, \gamma > 0$

$$\begin{aligned}
S^{1,q;\gamma} &= \inf_{a<0<b} \frac{\int_a^b |x|^\gamma\,dS}{(\int_a^b |x|^{q\gamma-1}\,dx)^{1/q}} \\
&= \inf_{a<0<b} \frac{(|a|^\gamma + |b|^\gamma)(q\gamma)^{1/q}}{(|a|^{q\gamma} + |b|^{q\gamma})^{1/q}} = (q\gamma)^{1/q}.
\end{aligned} \tag{3.7}$$

Thus we see that $S^{1,q;\gamma}_{\mathrm{rad}} = 2^{1-1/q} S^{1,q;\gamma} > S^{1,q;\gamma}$. □

Remark 3.1. In the estimate (3.5), it suffices to assume that $\gamma \geq C(q,n)\delta^{(n-1)(q-1^*)}$, where $C(q,n)$ are some positive numbers depending only on q and n.

Proof of Theorem 1.5.

1. We assume $0 < \gamma < \overline{\gamma}$ and $S^{1,q;\overline{\gamma}} = S^{1,q;\overline{\gamma}}_{\mathrm{rad}}$, then it follows from Theorem 1.3;2 and Proposition 2.1;1 that $S^{1,q;\gamma}_{\mathrm{rad}} \leq S^{1,q;\gamma}$, and this proves the assertion.
2. There exists a nonradial function $u \in W^{1,1}_{\gamma,0}(\mathbf{R}^n)$ such that we have for any $\varepsilon > 0$

$$S^{1,q;\overline{\gamma}} - \varepsilon \leq E^{1,q;\overline{\gamma}}(u) \leq S^{1,q;\overline{\gamma}} + \varepsilon. \tag{3.8}$$

Assume that $\gamma > \overline{\gamma}$, then we have

$$\begin{aligned}
S^{1,q;\gamma} &\leq E^{1,q;\gamma}(T_{\gamma/\overline{\gamma}}u) < (\overline{\gamma}/\gamma)^{\tau_{1,q}-1} E^{1,q;\overline{\gamma}}(u) \\
&\qquad \text{(by Lemma 2.1 with } \beta = \overline{\gamma}/\gamma) \\
&\leq (\overline{\gamma}/\gamma)^{\tau_{1,q}-1} S^{1,q;\overline{\gamma}} + (\overline{\gamma}/\gamma)^{\tau_{1,q}-1}\varepsilon \\
&\leq (\overline{\gamma}/\gamma)^{\tau_{1,q}-1} S^{1,q;\overline{\gamma}}_{\mathrm{rad}} + (\overline{\gamma}/\gamma)^{\tau_{1,q}-1}\varepsilon \\
&= S^{1,q;\gamma}_{\mathrm{rad}} + (\overline{\gamma}/\gamma)^{\tau_{1,q}-1}\varepsilon.
\end{aligned}$$

Hence we see $S^{1,q;\gamma} < S^{1,q;\gamma}_{\mathrm{rad}}$ by $\varepsilon \to 0$. □

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References

[1] B. Abdellaoui, E. Colorado and I. Peral, Some improved Caffarelli-Kohn-Nirenberg inequalities, Calc. Var. Partial Differential Equations **23** (2005), no. 3, 327–345.

[2] Y. Bozhkov, A Caffarelli-Kohn-Nirenberg type inequality on Riemannian manifolds, Appl. Math. Lett. **23** (2010), no. 10, 1166–1169.

[3] J. Byeon and Z.-Q. Wang, Symmetry breaking of extremal functions for the Caffarelli-Kohn-Nirenberg inequalities, Commun. Contemp. Math. **4** (2002), no. 3, 457–465.

[4] L. Caffarelli, R. Kohn and L. Nirenberg, First order interpolation inequalities with weights, Compositio Math. **53** (1984), no. 3, 259–275.

[5] F. Catrina and Z.-Q. Wang, On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions, Comm. Pure Appl. Math. **54** (2001), no. 2, 229–258.

[6] F. Catrina and Z.-Q. Wang, Positive bound states having prescribed symmetry for a class of nonlinear elliptic equations in $\mathbf{R}^N$, Ann. Inst. H. Poincaré Anal. Non Linéaire **18** (2001), no. 2, 157–178.

[7] N. Chiba and T. Horiuchi, On radial symmetry and its breaking in the Caffarelli-Kohn-Nirenberg type inequalities, Math. J. Ibaraki Univ. **47** (2015), 49–63.

[8] K. S. Chou and C. W. Chu, On the best constant for a weighted Sobolev-Hardy inequality, J. London Math. Soc. (2) **48** (1993), no. 1, 137–151.

[9] A. Cianchi and N. Fusco, Functions of bounded variation and rearrangements, Arch. Ration. Mech. Anal. **165** (2002), no. 1, 1–40.

[10] D. G. Costa, Some new and short proofs for a class of Caffarelli-Kohn-Nirenberg type inequalities, J. Math. Anal. Appl. **337** (2008), no. 1, 311–317.

[11] J. Dolbeault and M. J. Esteban, A scenario for symmetry breaking in Caffarelli-Kohn-Nirenberg inequalities, J. Numer. Math. **20** (2012), no. 3–4, 233–249.

[12] J. Dolbeault and M. J. Esteban, Extremal functions for Caffarelli-Kohn-Nirenberg and logarithmic Hardy inequalities, Proc. Roy. Soc. Edinburgh Sect. A **142** (2012), no. 4, 745–767.

[13] J. Dolbeault and M. J. Esteban, About existence, symmetry and symmetry breaking for extremal functions of some interpolation functional inequalities, in *Nonlinear partial differential equations*, Abel Symposia, 7, Springer, Heidelberg, 2012, pp. 117–130.

[14] J. Dolbeault, M. J. Esteban, M. Loss and G. Tarantello, On the symmetry of extremals for the Caffarelli-Kohn-Nirenberg inequalities, Adv. Nonlinear Stud. **9** (2009), no. 4, 713–726.

[15] J. Dolbeault, M. J. Esteban and G. Tarantello, The role of Onofri type inequalities in the symmetry properties of extremals for Caffarelli-Kohn-Nirenberg inequalities, in two space dimensions, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **7** (2008), no. 2, 313–341.

[16] J. Dolbeault, M. J. Esteban, G. Tarantello and A. Tertikas, Radial symmetry and symmetry breaking for some interpolation inequalities, Calc. Var. Partial Differential Equations **42** (2011), no. 3–4, 461–485.

[17] V. Felli and M. Schneider, Perturbation results of critical elliptic equations of Caffarelli-Kohn-Nirenberg type, J. Differential Equations **191** (2003), no. 1, 121–142.

[18] J. García-Azorero, I. Peral and A. Primo, A borderline case in elliptic problems involving weights of Caffarelli-Kohn-Nirenberg type, Nonlinear Anal. **67** (2007), no. 6, 1878–1894.

[19] T. Horiuchi, The imbedding theorems for weighted Sobolev spaces, J. Math. Kyoto Univ. **29** (1989), no. 3, 365–403.

[20] T. Horiuchi, Best constant in weighted Sobolev inequality with weights being powers of distance from the origin, J. Inequal. Appl. **1** (1997), no. 3, 275–292.

[21] T. Horiuchi, The symmetry breaking of the non-critical Caffarelli-Kohn-Nirenberg type inequalities by a linerization method, RIMS Kôkyûroku, **1810** (2012), 113–138.

[22] T. Horiuchi and P. Kumlin, On the Caffarelli-Kohn-Nirenberg-type inequalities involving critical and supercritical weights, Kyoto J. Math. **52** (2012), no. 4, 661–742.

[23] C.-S. Lin and Z.-Q. Wang, Symmetry of extremal functions for the Caffarrelli-Kohn-Nirenberg inequalities, Proc. Amer. Math. Soc. **132** (2004), no. 6, 1685–1691 (electronic).

[24] V. G. Maz'ja, *Sobolev spaces*, translated from the Russian by T. O. Shaposhnikova, Springer Series in Soviet Mathematics, Springer, Berlin, 1985.

[25] G. Talenti, Best constant in Sobolev inequality, Ann. Mat. Pura Appl. (4) **110** (1976), 353–372.

[26] Z.-Q. Wang and M. Willem, Caffarelli-Kohn-Nirenberg inequalities with remainder terms, J. Funct. Anal. **203** (2003), no. 2, 550–568.