Asymptotic behavior of Lévy measure density corresponding to inverse local time

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Abstract: For a one dimensional diffusion process $\mathbf{D}^*_{s,m}$ and the harmonic transformed process $\mathbf{D}^*_{s_h,m_h}$, the asymptotic behavior of the Lévy measure density corresponding to the inverse local time at the regular end point is investigated. The asymptotic behavior of n^*, the Lévy measure density corresponding to $\mathbf{D}^*_{s,m}$, follows from asymptotic behavior of the speed measure m. However, that of n^{h*}, the Lévy measure density corresponding to $\mathbf{D}^*_{s_h,m_h}$, is given by a simple form, n^* multiplied by an exponential decay function, for any harmonic function h based on the original diffusion operator.

Key words: Lévy measure density; asymptotic behavior; inverse local time.

1. Inverse local time and Lévy measure density. Let s be a continuous increasing function on an open interval $I=(l_1,l_2)$, where $-\infty<l_1<l_2\le\infty$, and let m be a right continuous increasing function on I. We assume

(1) $$|s(l_1)|+|m(l_1)|<\infty,$$

where we set $u(l_i)=\lim_{x\to l_i, x\in I}u(x)$, $i=1,2$, if there exist the limits, for functions u on I. (1) implies that the end point l_1 is regular in the sense of Feller [2]. We pose the *reflecting* or *absorbing* boundary condition at l_i $(i=1,2)$ if it is regular. Let $\mathcal{G}_{s,m}$ be a one dimensional diffusion operator on I with scale function s, speed measure m, and null killing measure. We denote by $\mathbf{D}^*_{s,m}=[X(t),P^*_x]$ [resp. $\mathbf{D}^o_{s,m}=[X(t),P^o_x]$] the one dimensional diffusion process on I with $\mathcal{G}_{s,m}$ as the generator and with l_1 being reflecting [resp. absorbing]. Let denote by $l^*(t,\xi)$ the local time of $\mathbf{D}^*_{s,m}$, that is,

$$\int_0^t f(X(u))\,du=\int_I l^*(t,\xi)f(\xi)\,dm(\xi),\quad t>0,$$

for bounded continuous functions f on I. Since $l^*(t,\xi)$ is continuous and nondecreasing in t P^*_x-a.s., there is the right continuous inverse function $l^{*-1}(t,\xi)$. Note that there exists the inverse local time $l^{*-1}(t,l_1)$ at the end point l_1, which is denoted by $\tau^*(t)$. Combining Lévy formulas due to R. M. Blumenthal and R. K. Getoor ([1], Chapter V, Theorem 3.21) and those due to K. Itô and H. P. McKean ([3], Section 6.2), we obtain the following result. We give the proof in another paper.

Proposition 1. *The Laplace transform of the distribution of* $[\tau^*(t),\ t\ge 0]$ *is given by the following*

(2) $$E^*_{l_1}[e^{-\lambda\tau^*(t)}]=\exp\Big\{-\gamma^*t-t\int_0^\infty(1-e^{-\lambda\xi})n^*(\xi)\,d\xi\Big\},$$

(3) $$\gamma^*=\begin{cases}0 & \text{if } s(l_2)=\infty,\ \text{or}\\ & l_2 \text{ is regular and reflecting},\\ 1/\{s(l_2)-s(l_1)\} & \text{if } s(l_2)<\infty,\end{cases}$$

(4) $$n^*(\xi)=\lim_{x\to l_1}q^*(\xi,x)/\{s(x)-s(l_1)\},$$

where $E^*_{l_1}$ *stands for the expectation with respect to* $P^*_{l_1}$,

(5) $$\int_0^t q^*(\xi,x)\,d\xi=P^*_x(\sigma_{l_1}<t),\quad x\in I,\ t>0,$$

and σ_{l_1} *is the first hitting time for* l_1.

We note a representation of $q^*(\xi,x)$ in terms of the transition probability density $p^o(t,x,y)$ of $\mathbf{D}^o_{s,m}$, that is,

(6) $$q^*(\xi,x)=\lim_{z\to l_1}p^o(\xi,z,x)/\{s(z)-s(l_1)\},\quad \xi>0,\ x\in I.$$

Here $p^o(t,x,y)$ is the transition probability density with respect to dm for $\mathbf{D}^o_{s,m}$, that is,

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$$P_x^o(X(t)\in E)=\int_E p^o(t,x,y)\,dm(y),$$

for $x\in I$, $E\in\mathcal{B}(I)$, where $\mathcal{B}(I)$ stands for the set of all Borel sets of I. It is well known that $p^o(t,x,y)$ is represented as

$$(7)\quad p^o(t,x,y)=\int_{(0,\infty)} e^{-\lambda t}\psi^o(x,-\lambda)\psi^o(y,-\lambda)\,d\sigma^o(\lambda),\quad t>0,\ x,y\in I,$$

where $d\sigma^o(\lambda)$ is a Borel measure on $(0,\infty)$ satisfying

$$(8)\quad \int_{(0,\infty)} e^{-\lambda t}\,d\sigma^o(\lambda)<\infty,\quad t>0,$$

and $\psi^o(x,\alpha)$, $x\in I$, $\alpha\in\mathbf{C}$, is the unique solution of the following integral equation (9).

$$(9)\quad \psi^o(x,\alpha)=s(x)-s(l_1)+\alpha\int_{(l_1,x]}\{s(x)-s(y)\}\psi^o(y,\alpha)\,dm(y).$$

By means of (4), (6), (7) and (9), we find

$$(10)\quad n^*(\xi)=\lim_{x,y\to l_1} D_{s(x)}D_{s(y)}p^o(\xi,x,y)=\int_{(0,\infty)} e^{-\lambda\xi}d\sigma^o(\lambda),$$

where $D_{s(x)}$ denotes the right derivative with respect to $s(x)$. $n^*(\xi)$ is the Lévy measure density of the inverse local time $[\tau^*(t),\ t\geq 0]$.

Example 2. Let $l_1=0$, $l_2=l<\infty$, $s(x)=x$ and $m(x)=C(l-x)^{-(1+1/\rho)}$, where C is a positive number and $0<\rho<1$. (1) is satisfied. By virtue of Proposition 1, the Laplace transform of the distribution of $[\tau^*(t),\ t\geq 0]$ is given by

$$(11)\quad E_0^*[e^{-\lambda\tau^*(t)}]=\exp\left\{-t/l-t\int_0^\infty(1-e^{-\lambda\xi})n^*(\xi)\,d\xi\right\},$$

$$(12)\quad n^*(\xi)=\int_0^\infty e^{-\lambda t}\sigma_o^o(\lambda)\,d\lambda,$$

where

$$(13)\quad p^o(t,x,y)=\int_{(0,\infty)} e^{-\lambda t}\psi^o(x,-\lambda)\psi^o(y,-\lambda)\,\sigma_o^o(\lambda)\,d\lambda,$$

$$(14)\quad \psi^o(x,-\lambda)=\rho\pi\sqrt{l(l-x)}\times\{-N_\rho(c_\rho l^{-1/2\rho}\sqrt{\lambda})J_\rho(c_\rho(l-x)^{-1/2\rho}\sqrt{\lambda})+J_\rho(c_\rho l^{-1/2\rho}\sqrt{\lambda})N_\rho(c_\rho(l-x)^{-1/2\rho}\sqrt{\lambda})\},$$

$$(15)\quad \sigma_o^o(\lambda)=(l\rho\pi^2)^{-1}\times\{J_\rho(c_\rho l^{-1/2\rho}\sqrt{\lambda})^2+N_\rho(c_\rho l^{-1/2\rho}\sqrt{\lambda})^2\}^{-1}.$$

Here $c_\rho=2\{C\rho(1+\rho)\}^{1/2}$, and $J_\rho(z)$ are $N_\rho(z)$ are Bessel functions. We prove (13) with (14) and (15) in another paper. Noting the asymptotic behavior of Bessel functions, we have

$$\sigma_o^o(\lambda)\sim\frac{C^\rho C_1(\rho)}{l^2\Gamma(1+\rho)}\lambda^\rho\quad\text{as}\quad\lambda\to 0,$$

where $f(t)\sim g(t)$ as $t\to 0$ [resp. $t\to\infty$] stands for $\lim_{t\to 0\,[\text{resp. } t\to\infty]} f(t)/g(t)=1$ for positive functions $f(t)$ and $g(t)$, and $C_1(\rho)$ is a positive number given by

$$(16)\quad C_1(\rho)=\{\rho(1+\rho)\}^\rho/\Gamma(\rho).$$

Therefore we find

$$(17)\quad n^*(\xi)\sim l^{-2}C^\rho C_1(\rho)\xi^{-(1+\rho)}\quad\text{as}\quad\xi\to\infty.$$

Example 3. Let $l_1=0$, $l_2=\infty$, $s(x)=x$ and $m(x)=Cx^{-1+1/\rho}$, where C is a positive number and $0<\rho<1$. (1) is satisfied. By virtue of Proposition 1, the Laplace transform of the distribution of $[\tau^*(t),\ t\geq 0]$ is given by

$$(18)\quad E_0^*[e^{-\lambda\tau^*(t)}]=\exp\left\{-t\int_0^\infty(1-e^{-\lambda\xi})n^*(\xi)\,d\xi\right\},$$

$$(19)\quad n^*(\xi)=\int_0^\infty e^{-\lambda t}\sigma_o^o(\lambda)\,d\lambda=C^\rho C_2(\rho)\xi^{-(1+\rho)},$$

where $C_2(\rho)$ is a positive number given by

$$(20)\quad C_2(\rho)=\{\rho(1-\rho)\}^\rho/\Gamma(\rho).$$

(19) follows from the following representation of $p^o(t,x,y)$.

$$(21)\quad p^o(t,x,y)=\int_{(0,\infty)} e^{-\lambda t}\psi^o(x,-\lambda)\psi^o(y,-\lambda)\,\sigma_o^o(\lambda)\,d\lambda,$$

$$(22)\quad \psi^o(x,-\lambda)=\frac{\Gamma(1+\rho)}{\{C\rho(1-\rho)\lambda\}^{\rho/2}}\sqrt{x}\times J_\rho(2\sqrt{C\rho(1-\rho)\lambda}x^{1/2\rho}),$$

$$(23)\quad \sigma_o^o(\lambda)=\frac{C^\rho C_2(\rho)}{\Gamma(1+\rho)}\lambda^\rho.$$

2. Asymptotic behavior of Lévy measure densities. In this section we consider asymptotic behavior of Lévy measure densities. We assume one of the following (**A1**), (**A2**) and (**A3**), where $0 < \rho < 1$ and $L(x)$ is a slowly varying function.

(**A1**) $l_1 = 0,\ l_2 = l < \infty,\ s(x) = x$ and $m(x)$ satisfies $|m(0)| < \infty$ and

(24) $$m(l - 1/x) \sim x^{1+1/\rho} L(x) \quad \text{as} \quad x \to \infty.$$

(**A2**) $l_1 = 0,\ l_2 = \infty,\ s(x) = x$ and $m(x)$ satisfies $|m(0)| < \infty$ and

(25) $$m(x) \sim x^{-1+1/\rho} L(x) \quad \text{as} \quad x \to \infty.$$

(**A3**) $l_1 = 0,\ l_2 = \infty,\ s(x) = x$ and $m(x)$ satisfies $\lim_{x\to\infty} m(x) = \infty$ and

(26) $$m(x) \sim x^{-1+1/\rho} L(x) \quad \text{as} \quad x \to 0.$$

Since $l_1 = 0$ is regular, we can define the inverse local time $\tau^*(t)$ at 0 by putting the reflecting boundary condition. We obtain the following asymptotic behavior of Lévy measure densities. Let $K(x)$ be another slowly varying function such that

(27) $$\lim_{x\to\infty} K(x)^{1/\rho} L(x^\rho K(x)) = \lim_{x\to\infty} L(x)^\rho K(x^{1/\rho} L(x)) = 1,$$

where $x \to \infty$ should be read as $x \to 0$ when (**A3**) is satisfied.

Theorem 4. *Assume* (**A1**). *Then the Laplace transform of the distribution of* $[\tau^*(t),\ t \geq 0]$ *is given by the same formula as* (11) *and the Lévy measure density satisfies*

(28) $$n^*(\xi) \sim l^{-2} C_1(\rho) \xi^{-(1+\rho)} K(\xi)^{-1} \quad \textit{as} \quad \xi \to \infty.$$

Theorem 5. *Assume* (**A2**) [*resp.* (**A3**)]. *Then the Laplace transform of the distribution of* $[\tau^*(t),\ t \geq 0]$ *is given by the same formula as* (18) *and the Lévy measure density satisfies*

(29) $$n^*(\xi) \sim C_2(\rho) \xi^{-(1+\rho)} K(\xi)^{-1} \quad \textit{as} \quad \xi \to \infty \quad [\textit{resp.}\ \xi \to 0].$$

Proof of Theorem 4. The assumption (**A1**) implies that (A.1) with $\theta = 0$ of [10] is satisfied, where we should replace the role of l_1 by that l_2 in (A.1) of [10]. Since $l_1 = 0$ is regular, we can put $l_1 = 0$ in (3.1) of [10]. Thus, by means of (5.11) of [10], we have

$$\int_{(0,\infty)} e^{-\lambda t}\, d\sigma^o(\lambda) \sim l^{-2} C_1(\rho) t^{-(1+\rho)} K(t)^{-1} \quad \text{as} \quad t \to \infty.$$

Combining this with (10), we obtain (28). □

Theorem 5 follows from some results on Krein's correspondence. The arguments of Krein's correspondence are due to [4] and [6]. Let denote by $\mathcal{M}$ the totality of nonnegative right continuous nondecreasing functions $\mu(x)$ on $[0, \infty]$ such that $\mu(x) \not\equiv \infty$ and $\mu(\infty) = \infty$. For $\mu \in \mathcal{M}$ set $\mu(0-) = 0$ and let $\varphi(x, \lambda)$ be the solution of the integral equation

$$\varphi(x, \lambda) = 1 + \lambda \int_{[0,x]} (x - y) \varphi(y, \lambda)\, d\mu(y), \quad x \in [0, l),$$

where $\lambda \in \mathbf{C}$ and $l = \sup\{x : \mu(x) < \infty\}$. We set

$$\kappa(\alpha) = \int_0^l \varphi(x, \alpha)^{-2}\, dx, \quad \alpha > 0.$$

κ is called the *characteristic function* of μ and the correspondence $\mu \in \mathcal{M} \to \kappa$ is called *Krein's correspondence.* Let $\mathcal{K}$ be the set of functions κ on $(0, \infty)$ such that

$$\kappa(\alpha) = c + \int_{[0,\infty)} (\alpha + \lambda)^{-1}\, d\sigma(\lambda), \quad \alpha > 0,$$

for some $c \geq 0$ and some nonnegative Borel measure σ on $[0, \infty)$ satisfying $\int_{[0,\infty)} (1 + \lambda)^{-1}\, d\sigma(\lambda) < \infty$. It is well known that Krein's correspondence is a one to one map from $\mathcal{M}$ onto $\mathcal{K}$ (see [4], e.g.). From now on we denote by $\mu \in \mathcal{M} \leftrightarrow \kappa \in \mathcal{K}$ Krein's correspondence. In [5] Kasahara proved the following asymptotic theorem on Krein's corespondence, where $0 < \rho < 1$, $L(x)$ and $K(x)$ are slowly varying functions satisfying (27), and $C_3(\rho) = \rho / \{\Gamma(1 - \rho) C_2(\rho)\}$.

Theorem 6 ([5])**.** $\mu \in \mathcal{M} \leftrightarrow \kappa \in \mathcal{K}$ *and* $l = \infty$. *Then the following* (30), (31) *and* (32) *are equivalent each other.*

(30) $$\mu(x) \sim x^{-1+1/\rho} L(x) \quad \textit{as } x \to \infty\ [x \to 0].$$

(31) $$\kappa(\alpha) \sim C_3(\rho) \alpha^{-\rho} K(1/\alpha) \quad \textit{as } \alpha \to 0\ [\alpha \to \infty].$$

(32) $$\sigma(\lambda) \sim \{C_3(\rho)/\Gamma(\rho)\Gamma(2 - \rho)\} \lambda^{1-\rho} K(1/\lambda) \quad \textit{as} \quad \lambda \to 0\ [\lambda \to \infty].$$

Now we show Theorem 5.

Proof of Theorem 5. Assume (**A2**) [resp. (**A3**)]. Since $m \in \mathcal{M}$, there is the characteristic function $\kappa \in \mathcal{K}$ such that $m \leftrightarrow \kappa$. By means of Theorem 6, $\kappa(\alpha)$ satisfies (31). As we saw in Lemma 3 of [7], $1/\alpha\kappa(\alpha) \in \mathcal{K}$ and the corresponding spectral measure $d\sigma_*(\lambda)$ coincides with $d\sigma^o(\lambda)/\lambda$ for $\lambda > 0$ and $\sigma_*(\{0\}) = 0$. Noting $1/\alpha\kappa(\alpha) \sim$

$C_3(\rho)^{-1}\alpha^{\rho-1}/K(1/\alpha)$ as $\alpha \to 0$ [resp. $\alpha \to \infty$], and the relation between (31) and (32), we get

$$\sigma_*(\lambda) \sim \{\Gamma(1-\rho)\Gamma(1+\rho)C_3(\rho)\}^{-1}\lambda^\rho/K(1/\lambda) \quad \text{as } \lambda \to 0 \text{ [resp. } \lambda \to \infty],$$

and hence

$$\sigma^o(\lambda) = \int_{(0,\lambda]} \xi\, d\sigma_*(\xi) \sim \frac{\rho}{1+\rho}\lambda\sigma_*(\lambda) \quad \text{as } \lambda \to 0 \text{ [resp. } \lambda \to \infty].$$

Thus we obtain

$$\int_{(0,\infty)} e^{-\lambda t}\, d\sigma^o(\lambda) \sim C_2(\rho)t^{-(1+\rho)}K(t)^{-1} \quad \text{as} \quad t \to \infty \text{ [resp. } t \to 0].$$

Combining this with (10), we obtain (29). □

3. Inverse local time of harmonic transformed diffusion processes. In this section we consider inverse local times of harmonic transformed diffusion processes and the corresponding Lévy measure densities. Let $\mathbf{D}^*_{s,m}$ and $\mathbf{D}^o_{s,m}$ be diffusion processes on I as in Section 1. For both diffusion processes we pose the *absorbing* boundary condition at l_2 whenever it is regular, that is, $|s(l_2)| + |m(l_2)| < \infty$.

For $\beta \geq 0$, let h be a positive continuous function on I satisfying $\mathcal{G}_{s,m}h = \beta h$. We set

$$s_h(x) = \int_{(c_0,x]} h(y)^{-2}\, ds(y),$$
$$m_h(x) = \int_{(c_0,x]} h(y)^2\, dm(y),$$

where $c_0 \in I$ is fixed arbitrarily. Let us consider a harmonic transformed diffusion process on I whose generator is given by $\mathcal{G}_{s_h,m_h}$. It is known that $h(x)$ is represented as a linear combination of $g_i(x,\beta)$ ($i = 1,2$) such that $g_i(x,\beta)$ is positive and continuous in x, $g_1(x,\beta)$ is nondecreasing in x, $g_2(x,\beta)$ is nonincreasing in x, $g_i(l_i,\beta) = 0$ if $|s(l_i)| < \infty$, and $\mathcal{G}_{s,m}g_i = \beta g_i$. Note that there exist such functions $g_i(,\beta)$, $i = 1,2$ ([3]). In the following we set

$$h(x) = B_1 g_1(x,\beta) + B_2 g_2(x,\beta), \tag{33}$$

where $B_1 \geq 0$, $B_2 > 0$. Since $g_1(l_1,\beta) = 0$, (33) implies $h(l_1) \in (0,\infty)$, and by virtue of Theorem 1.1 of [8], $|s_h(l_1)| + |m_h(l_1)| < \infty$, that is, l_1 is regular for harmonic transformed diffusion processes. Let $\mathbf{D}^*_{s_h,m_h} = [X(t), P^{h*}_x]$ [resp. $\mathbf{D}^o_{s_h,m_h} = [X(t), P^{ho}_x]$] the one dimensional diffusion process on I with $\mathcal{G}_{s_h,m_h}$ as the generator and with l_1 being reflecting [resp. absorbing]. For both diffusion processes we pose the *absorbing* boundary condition at l_2 whenever it is regular, that is, $|s_h(l_2)| + |m_h(l_2)| < \infty$. We denote by $[\tau^{(h*)}(t),\, t \geq 0]$ the inverse local time of $\mathbf{D}^*_{s_h,m_h}$ at the end point l_1.

We derive the following result from Proposition 1, Theorem 1.1 of [8] and Theorem 3.2 of [9].

Theorem 7. *The Laplace transform of the distribution of* $[\tau^{h*}(t),\, t \geq 0]$ *is given by the following*

$$E^{h*}_{l_1}\Big[e^{-\lambda\tau^{h*}(t)}\Big] = \exp\Big\{-\gamma^{h*}t - t\int_0^\infty (1-e^{-\lambda\xi})n^{h*}(\xi)\, d\xi\Big\}, \tag{34}$$

$$\gamma^{h*} = \begin{cases} 0 & \text{if } B_1 = 0, \\ 1/\{s_h(l_2) - s_h(l_1)\} & \text{if } B_1 > 0, \end{cases} \tag{35}$$

$$n^{h*}(\xi) = (B_2 g_2(l_1,\beta))^2 e^{-\beta\xi} n^*(\xi), \tag{36}$$

where $E^{h*}_{l_1}$ *stands for the expectation with respect to* $P^{h*}_{l_1}$ *and* $n^*(\xi)$ *is given by* (4).

We should note that n^{h*} is independent of B_1.

Finally we study asymptotic behavior of Lévy measure density $n^{h*}(\xi)$. Assume that $\mathbf{D}^*_{s,m}$ satisfies one of (**A1**), (**A2**) and (**A3**). We might suppose that the asymptotic behavior of $n^{h*}(\xi)$ depends on those of $s_h(x)$ and $m_h(x)$ as $x \to l_2$, and hence that of $h(x)$ as $x \to l_2$. However the asymptotic behavior of $n^{h*}(\xi)$ is given by a quite simple form $n^*(\xi)$ multiplied by $e^{-\beta\xi}$.

Theorem 8. *Assume one of* (**A1**), (**A2**) *and* (**A3**). *Let h be given by* (33). *Then* (34), (35) *and* (36) *hold. In particular, the asymptotic behavior of Lévy measure density* n^{h*} *is given by* (36) *with* $n^*(\xi)$ *satisfying* (28) [*resp.* (29)] *if* (**A1**) [*resp.* (**A2**) *or* (**A3**)] *is satisfied.*

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