Generalized Bessel transform of (β,γ)-generalized Bessel Lipschitz functions

Dedicated to Professor François Rouvière for 70th birthday

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Abstract: In this paper, we prove an analog of Younis's theorem 5.2 in [4] for the generalized Fourier-Bessel transform on the Half line for functions satisfying the (β,γ)-generalized Bessel Lipschitz condition in the space $L^2_{\alpha,n}$.

Key words: Generalized Fourier-Bessel transform; generalized translation operator.

1. Introduction and preliminaries. Younis ([4], Theorem 5.2) characterized the set of functions in $L^2(\mathbf{R})$ satisfying the Dini-Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transforms, namely we have

Theorem 1.1 ([4], Theorem 5.2). *Let* $f \in L^2(\mathbf{R})$. *Then the following are equivalents*:

1. $\|f(.+h) - f(.)\|_{L^2(\mathbf{R})} = O(\frac{h^\alpha}{(\log\frac{1}{h})^\beta})$ *as* $h \to 0, 0 < \alpha < 1, \beta > 0$,
2. $\int_{|\lambda|\geq r} |\mathcal{F}(f)(\lambda)|^2 d\lambda = O(r^{-2\alpha}(\log r)^{-2\beta})$ *as* $r \to +\infty$,

where $\mathcal{F}$ *stands for the Fourier transform of* f.

The main aim of this paper is to establish an analog of Theorem 1.1 in the generalized Fourier-Bessel transform. We point out that similar results have been established in the Dunkl transform [3].

We briefly overview the theory of generalized Fourier-Bessel transform and related harmonic analysis (see [2]).

Consider the second-order singular differential operator on the half line

$$\mathrm{B}f(x) = \frac{d^2 f(x)}{dx^2} + \frac{(2\alpha+1)}{x}\frac{df(x)}{dx} - \frac{4n(\alpha+n)}{x^2} f(x),$$

where $\alpha > -\frac{1}{2}$ and $n = 0, 1, 2, \ldots$. For $n = 0$, we obtain the classical Bessel operator

$$\mathrm{B}_\alpha f(x) = \frac{d^2 f(x)}{dx^2} + \frac{(2\alpha+1)}{x}\frac{df(x)}{dx}.$$

For $\alpha > -\frac{1}{2}$ and $n = 0, 1, 2, \ldots$, let M be the map defined by

$$\mathrm{M}f(x) = x^{2n} f(x).$$

Let $L^2_{\alpha,n}$ be the class of measurable functions f on $[0, \infty[$ for which

$$\|f\|_{2,\alpha,n} = \|\mathrm{M}^{-1}f\|_{2,\alpha+2n} < \infty,$$

where

$$\|f\|_{2,\alpha} = \left(\int_0^{+\infty} |f(x)|^2 x^{2\alpha+1} dx\right)^{1/2}.$$

For $\alpha > -\frac{1}{2}$, we introduce the normalized spherical Bessel function j_α defined by

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+p+1)}\left(\frac{z}{2}\right)^{2k}, z \in \mathbf{C},$$

where $\Gamma(x)$ is the gamma-function. The function $y = j_\alpha(x)$ satisfies the differential equation

$$\mathrm{B}_\alpha y + y = 0$$

with the initial conditions $y(0) = 1$ and $y'(0) = 0$. The function $j_\alpha(x)$ is infinitely differentiable, even, and, moreover entire analytic.

Lemma 1.2. *For* $x \in \mathbf{R}^+$ *the following inequalities are fulfilled.*

1. $|j_\alpha(x)| \leq 1$,
2. $|1 - j_\alpha(x)| \leq x$,
3. $|1 - j_\alpha(x)| \geq c$ *with* $x \geq 1$, *where* $c > 0$ *is a certain constant which depends only on* α.

Proof (See [1]). □

For $\lambda \in \mathbf{C}$ and $x \in \mathbf{R}$, put

$$\varphi_\lambda(x) = x^{2n} j_{\alpha+2n}(\lambda x).$$

From [2] recall the following properties.

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Proposition 1.3. 1. *φ_λ satisfies the differential equation*

$$B\varphi_\lambda = -\lambda^2\varphi.$$

2. *For all $\lambda \in \mathbf{C}$, and $x \in \mathbf{R}$*

$$|\varphi_\lambda(x)| \le x^{2n} e^{|Im\lambda||x|}.$$

The generalized Fourier-Bessel transform we call the integral from [2]

$$\mathcal{F}_B(f)(\lambda) = \int_0^{+\infty} f(x)\varphi_\lambda(x)x^{2\alpha+1}dx, \lambda \ge 0,\ f \in L^1_{\alpha,n}.$$

Let $f \in L^1_{\alpha,n}$, the inverse generalized Fourier-Bessel transform is given by the formula

$$f(x) = \int_0^{+\infty} \mathcal{F}_B(f)(\lambda)\varphi_\lambda(x)d\mu_{\alpha+2n}(\lambda),$$

where

$$d\mu_{\alpha+2n}(\lambda) = \frac{1}{4^{\alpha+2n}(\Gamma(\alpha+2n+1))^2}\lambda^{2\alpha+4n+1}d\lambda.$$

From [2], we have

Theorem 1.4. 1. *For every $f \in L^1_{\alpha,n} \cap L^2_{\alpha,n}$ we have the Plancherel formula*

$$\int_0^{+\infty} |f(x)|^2 x^{2\alpha+1}dx = \int_0^{+\infty} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

2. *The generalized Fourier-Bessel transform $\mathcal{F}_B$ extends uniquely to an isometric isomorphism from $L^2_{\alpha,n}$ onto $L^2([0,+\infty[,\mu_{\alpha+2n})$.*

Define the generalized translation operator T_h, $h > 0$ by the relation

$$T_h f(x) = (xh)^{2n}\tau^h_{\alpha+2n}(M^{-1}f)(x),\ x \ge 0,$$

where $\tau^h_{\alpha+2n}$ are the Bessel translation operators of order $\alpha+2n$ defined by

$$\tau^h_\alpha f(x) = c_\alpha \int_0^\pi f(\sqrt{x^2+h^2-2xh\cos t})\sin^{2\alpha} t dt,$$

where

$$c_\alpha = \left(\int_0^\pi \sin^{2\alpha} t dt\right)^{-1} = \frac{\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})}.$$

By Proposition 3.2 in [2], we have

$$\mathcal{F}_B(T_h f)(\lambda) = \varphi_\lambda(h)\mathcal{F}_B(f)(\lambda),\ f \in L^2_{\alpha,n}. \quad (1)$$

2. Main result. In this section we give the main result of this paper. We need first to define the (β,γ)-generalized Bessel Lipschitz class

Definition 2.1. Let $\beta \in (0,1)$ and $\gamma \ge 0$. A function $f \in L^2_{\alpha,n}$ is said to be in the (β,γ)-generalized Bessel Lipschitz class, denoted by $BLip(\beta,2,\gamma)$, if

$$\|T_h f(.) + T_{-h}f(.) - 2h^{2n}f(.)\|_{2,\alpha,n} = O\left(\frac{h^{\beta+2n}}{(\log\frac{1}{h})^\gamma}\right) \text{ as } h \to 0.$$

Theorem 2.2. *Let $f \in L^2_{\alpha,n}$. Then the following are equivalents*

1. *$f \in BLip(\beta,2,\gamma)$,*
2. *$\int_r^{+\infty} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O\left(\frac{r^{-2\beta}}{(\log r)^{2\gamma}}\right)$ as $r \to +\infty$.*

Proof. 1) $\Longrightarrow$ 2) Assume that $f \in BLip(\beta,2,\gamma)$. Then

$$\|T_h f(.) + T_{-h}f(.) - 2h^{2n}f(.)\|_{2,\alpha,n} = O\left(\frac{h^{\beta+2n}}{(\log\frac{1}{h})^\gamma}\right) \text{ as } h \to 0.$$

Formula (1), we have

$$\begin{aligned}\mathcal{F}_B(T_h f + T_{-h}f - 2h^{2n}f)(\lambda) &= (\varphi_\lambda(h) + \varphi_\lambda(-h) - 2h^{2n})\mathcal{F}_B(f)(\lambda)\\ &= (h^{2n}j_{\alpha+2n}(\lambda h) + (-h)^{2n}j_{\alpha+2n}(-\lambda h)\\ &\quad - 2h^{2n})\mathcal{F}_B(f)(\lambda)\\ &= 2h^{2n}(j_{\alpha+2n}(\lambda h) - 1)\mathcal{F}_B(f)(\lambda).\end{aligned}$$

Plancherel identity gives

$$\|T_h f(.) + T_{-h}f(.) - 2h^{2n}f(.)\|^2_{2,\alpha,n} = \int_0^{+\infty} 4h^{4n}|1 - j_{\alpha+2n}(\lambda h)|^2|\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

If $\lambda \in [\frac{1}{h},\frac{2}{h}]$, then $\lambda h \ge 1$ and (3) of Lemma 1.2 implies that

$$1 \le \frac{1}{c^2}|1 - j_{\alpha+2n}(\lambda h)|^2.$$

Then

$$\begin{aligned}&\int_{\frac{1}{h}}^{\frac{2}{h}} |\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\\ &\le \frac{1}{c^2}\int_{\frac{1}{h}}^{\frac{2}{h}} |1 - j_{\alpha+2n}(\lambda h)|^2|\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\\ &\le \frac{1}{c^2}\int_0^{+\infty} |1 - j_{\alpha+2n}(\lambda h)|^2|\mathcal{F}_B(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\end{aligned}$$

$$\leq \frac{1}{c^2}\frac{1}{4h^{4n}}\|\mathrm{T}_h f(.)+\mathrm{T}_{-h}f(.)-2h^{2n}f(.)\|^2_{2,\alpha,n} = O\left(\frac{h^{2\beta}}{(\log\frac{1}{h})^{2\gamma}}\right).$$

We obtain

$$\int_r^{2r}|\mathcal{F}_{\mathrm{B}}(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O\left(\frac{r^{-2\beta}}{(\log r)^{2\gamma}}\right) \ as\ r\to+\infty.$$

Thus these exists $C>0$ such that

$$\int_r^{2r}|\mathcal{F}_{\mathrm{B}}(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\leq C\frac{r^{-2\beta}}{(\log r)^{2\gamma}}.$$

So that

$$\begin{aligned}
&\int_r^{+\infty}|\mathcal{F}_{\mathrm{B}}(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\\
&= \left[\int_r^{2r}+\int_{2r}^{4r}+\int_{4r}^{8r}+\dots\right]|\mathcal{F}_{\mathrm{B}}(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\\
&\leq C\frac{r^{-2k}}{(\log r)^{2\gamma}}+C\frac{(2r)^{-2k}}{(\log 2r)^{2\gamma}}+C\frac{(4r)^{-2k}}{(\log 4r)^{2\gamma}}+\dots\\
&\leq C\frac{r^{-2k}}{(\log r)^{2\gamma}}(1+2^{-2k}+(2^{-2k})^2+(2^{-2k})^3+\dots)\\
&\leq CK\frac{r^{-2k}}{(\log r)^{2\gamma}},
\end{aligned}$$

where $K=(1-2^{-2k})^{-1}$ since $2^{-2k}<1$.

This prove that

$$\int_r^{+\infty}|\mathcal{F}_{\mathrm{B}}(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O\left(\frac{r^{-2k}}{(\log r)^{2\gamma}}\right) \ as\ r\to+\infty.$$

2) $\Longrightarrow$ 1) Suppose now that

$$\int_r^{+\infty}|\mathcal{F}_{\mathrm{B}}(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O\left(\frac{r^{-2k}}{(\log r)^{2\gamma}}\right) \ as\ r\to+\infty.$$

We write

$$\int_0^{+\infty}|1-j_{\alpha+2n}(\lambda h)|^2|\mathcal{F}_{\mathrm{B}}(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)=\mathrm{I}_1+\mathrm{I}_2,$$

where

$$\mathrm{I}_1=\int_0^{\frac{1}{h}}|1-j_{\alpha+2n}(\lambda h)|^2|\mathcal{F}_{\mathrm{B}}(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$$

and

$$\mathrm{I}_2=\int_{\frac{1}{h}}^{+\infty}|1-j_{\alpha+2n}(\lambda h)|^2|\mathcal{F}_{\mathrm{B}}(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

Estimate the summands I_1 and I_2.

From inequality (1) of Lemma 1.2, we have

$$\begin{aligned}
\mathrm{I}_2&=\int_{\frac{1}{h}}^{+\infty}|1-j_{\alpha+2n}(\lambda h)|^2|\mathcal{F}_{\mathrm{B}}(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\\
&\leq 4\int_{\frac{1}{h}}^{+\infty}|\mathcal{F}_{\mathrm{B}}(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\\
&= O\left(\frac{h^{2\beta}}{(\log\frac{1}{h})^{2\gamma}}\right).
\end{aligned}$$

Then

$$4h^{4n}\mathrm{I}_2=O\left(\frac{h^{2\beta+4n}}{(\log\frac{1}{h})^{2\gamma}}\right).$$

To estimate I_1, we use the inequality (2) of Lemma 1.2. Set

$$\phi(x)=\int_x^{+\infty}|\mathcal{F}_{\mathrm{B}}(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

Using integration by parts, we obtain

$$\begin{aligned}
\mathrm{I}_1&\leq -C_1h^2\int_0^{\frac{1}{h}}s^2\phi'(s)ds\\
&\leq -C_1\phi\left(\frac{1}{h}\right)+2C_1h^2\int_0^{\frac{1}{h}}s\phi(s)ds\\
&\leq C_2h^2\int_0^{1/h}ss^{-2\beta}(\log s)^{-2\gamma}ds\\
&\leq C_3h^{2\beta}\left(\log\frac{1}{h}\right)^{-2\gamma},
\end{aligned}$$

where C_1, C_2 and C_3 are positive constants.

Then

$$4h^{4n}\mathrm{I}_1=O\left(\frac{h^{2\beta+4n}}{(\log\frac{1}{h})^{2\gamma}}\right).$$

Furthermore, we have

$$\|\mathrm{T}_h f(.)+\mathrm{T}_{-h}f(.)-2h^{2n}f(.)\|_{2,\alpha,n} = O\left(\frac{h^{2\beta+4n}}{(\log\frac{1}{h})^{2\gamma}}\right) \ as\ h\to 0$$

and this ends the proof. □

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