Remark on Sturm bounds for Siegel modular forms of degree 2

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Abstract: We give Sturm bounds for Siegel modular forms of degree 2 with Fourier coefficients in an arbitrary algebraic number field K and for any prime ideal $\mathfrak{p}$ in K.

Key words: Siegel modular forms; congruences for modular forms; Fourier coefficients; J. Sturm.

1. Introduction. Sturm [8] studied how many Fourier coefficients we need, when we want to prove that an elliptic modular form vanishes modulo a prime ideal. We call such the numbers *Sturm bounds*. Poor-Yuen [7] studied initially Sturm bounds for Siegel modular forms of degree 2 having p-integral rational Fourier coefficients for any rational prime p. After their study, in [2], the author, Choi and Choie gave other type bounds with simple descriptions for them. However, the results in [2] are restricted to modular forms with Fourier coefficients in $\mathbf{Q}$ and a rational prime p with $p \neq 2,\ 3$. In this paper, we study the remaining parts, namely, we give such bounds for modular forms with Fourier coefficients in an arbitrary algebraic number field K and for any prime ideal $\mathfrak{p}$ in K.

2. Statement of results. In order to state our results, we fix notation. Let $\Gamma_2 = Sp_2(\mathbf{Z})$ be the Siegel modular group of degree 2 and $\mathbf{H}_2$ the Siegel upper-half space of degree 2. For a modular group Γ in Γ_2, we denote by $M_k(\Gamma)$ the $\mathbf{C}$-vector space of all Siegel modular forms of weight k for Γ.

Any f in $M_k(\Gamma)$ has a Fourier expansion of the form

$$f(Z) = \sum_{0 \le T \in \frac{1}{N}\Lambda_n} a_f(T) q^T,$$

$$q^T := e^{2\pi i\, \mathrm{tr}(TZ)}, \quad Z \in \mathbf{H}_2,$$

where T runs over all positive semi-definite elements of $\frac{1}{N}\Lambda_2$, N is the level of Γ and

$$\Lambda_2 := \{T = (t_{ij}) \in Sym_2(\mathbf{Q}) \mid t_{ii},\ 2t_{ij} \in \mathbf{Z}\}.$$

For simplicity, we write $T = (m, r, n)$ for $T = \begin{pmatrix} m & r/2 \\ r/2 & n \end{pmatrix} \in \frac{1}{N}\Lambda_2$ and also $a_f(m, r, n)$ for $a_f(T)$.

Let R be a subring of $\mathbf{C}$ and $M_k(\Gamma)_R \subset M_k(\Gamma)$ the R-module of all modular forms whose Fourier coefficients lie in R.

Let $f_1,\ f_2$ be two formal power series of the forms $f_i = \sum_{0 \le T \in \frac{1}{N}\Lambda_n} a_{f_i}(T) q^T$ with $a_{f_i}(T) \in R$. For an ideal I of R, we write

$$f_1 \equiv f_2 \mod I,$$

if and only if $a_{f_1}(T) \equiv a_{f_2}(T) \mod I$ for all $T \in \frac{1}{N}\Lambda_2$ with $T \ge 0$.

Let K be an algebraic number field and $\mathcal{O} = \mathcal{O}_K$ the ring of integers in K. For a prime ideal $\mathfrak{p}$ in $\mathcal{O}$, we denote by $\mathcal{O}_{\mathfrak{p}}$ the localization of $\mathcal{O}$ at $\mathfrak{p}$. Under these notation, we have

Theorem 2.1. (1) *Let k be even, $\mathfrak{p}$ an any prime ideal and $f \in M_k(\Gamma)_{\mathcal{O}_{\mathfrak{p}}}$ (with level N). Assume that $a_f(m, r, n) \equiv 0 \mod \mathfrak{p}$ for all $m,\ n \in \frac{1}{N}\mathbf{Z}$ with*

$$0 \le m, n \le \frac{k}{10}[\Gamma_2 : \Gamma]$$

and $r \in \frac{1}{N}\mathbf{Z}$ with $4mn - r^2 \ge 0$, then we have $f \equiv 0 \mod \mathfrak{p}$.
(2) *Let k be odd, $\mathfrak{p}$ an any prime ideal and $f \in M_k(\Gamma)_{\mathcal{O}_{\mathfrak{p}}}$ (with level N). Assume that $a_f(m, r, n) \equiv 0 \mod \mathfrak{p}$ for all $m,\ n \in \frac{1}{N}\mathbf{Z}$ with*

$$0 \le m, n \le \frac{k+35}{10}[\Gamma_2 : \Gamma]$$

and $r \in \frac{1}{N}\mathbf{Z}$ with $4mn - r^2 \ge 0$, then we have $f \equiv 0 \mod \mathfrak{p}$.

Remark 2.2. (1) For the case where k is odd, other type bounds were given in [3].
(2) Our bounds for the case of level 1 and even

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weight are sharp, because a similar argument in [2] works for the case $\mathfrak{p} \mid 2 \cdot 3$.

3. Proof of the theorem. In order to prove our theorem, we prepare two lemmas:

Lemma 3.1 (see [4] Lemma 3.7). *Let p be a prime. Assume that $\bigoplus_k M_k(\Gamma_n)_{\mathbf{Z}_{(p)}} = \mathbf{Z}_{(p)}[f_1, \cdots, f_s]$ with $f_i \in M_{k_i}(\Gamma_n)_{\mathbf{Z}_{(p)}}$. Then we have $\bigoplus_k M_k(\Gamma_n)_{\mathcal{O}_\mathfrak{p}} = \mathcal{O}_\mathfrak{p}[f_1, \cdots, f_s]$ for any prime ideal $\mathfrak{p}$ above p.*

Lemma 3.2. *Let p be a prime. Let W be the Witt operator. If $f \in M_k(\Gamma_2)_{\mathcal{O}_\mathfrak{p}}$ satisfies $W(f) = 0$, then $f/X_{10} \in M_{k-10}(\Gamma_2)_{\mathcal{O}_\mathfrak{p}}$, where $X_{10} \in S_{10}(\Gamma_2)_\mathbf{Z}$ is Igusa's cusp form of weight 10 normalized as $a_{X_{10}}(1,1,1) = 1$.*

Proof. Taking $q_{ij} := e^{2\pi i z_{ij}}$ with $Z = (z_{ij}) \in \mathbf{H}_2$, we can write

$$q^T = e^{2\pi i\,\mathrm{tr}(TZ)} = q_{12}^{2t_{12}} q_1^{t_1} q_2^{t_2},$$

where $q_i = q_{ii}$ and $t_i = t_{ii}$ $(i = 1, 2)$. Using this notation, we can regard $f \in M_k(\Gamma)_K$ as

$$f = \sum_{0 \le T \in \Lambda_n} a_f(T) q^T \in K[q_{12}^{-1}, q_{12}][\![q_1, q_2]\!].$$

Let $v_\mathfrak{p}$ be the normalized additive valuation with respect to $\mathfrak{p}$. We define a value $v_\mathfrak{p}(f)$ for any $f = \sum_{0 \le T \in \Lambda_n} a_f(T) q^T \in K[q_{12}^{-1}, q_{12}][\![q_1, q_2]\!]$ by

$$v_\mathfrak{p}(f) := \min\{v_\mathfrak{p}(a_f(T)) \mid T \in \Lambda_2\}.$$

It is easy to see that $v_\mathfrak{p}(fg) = v_\mathfrak{p}(f) + v_\mathfrak{p}(g)$ for any $f,\ g \in K[q_{12}^{-1}, q_{12}][\![q_1, q_2]\!]$. The assertion follows immediately from this fact and $v_\mathfrak{p}(X_{10}) = 0$. □

3.1. Proofs for the case of level 1.

Proof of (1) *for the case* $\mathfrak{p} \nmid 2 \cdot 3$. Combing these two lemmas with the Sturm type bounds for Jacobi forms obtained by [1], we can apply a similar argument of [2]. □

Proof of (1) *for the case* $\mathfrak{p} \mid 2 \cdot 3$. We denote by $\mathbf{F}$ the finite field defined by $\mathbf{F} = \mathcal{O}/\mathfrak{p}$. Let X_k $(k = 4, \cdots, 48)$ and Y_{12} be the fourteen generators of $\bigoplus_{k \in 2\mathbf{Z}} M_k(\Gamma_2)_\mathbf{Z}$ given by Igusa [5]. It is clear that, for $p = 2$ and 3, the graded algebra $\bigoplus_{k \in 2\mathbf{Z}} M_k(\Gamma_2)_{\mathbf{Z}_{(p)}}$ over $\mathbf{Z}_{(p)}$ is generated by these fourteen generators. Hence, for a prime ideal $\mathfrak{p}$ above p, $\bigoplus_{k \in 2\mathbf{Z}} M_k(\Gamma_2)_{\mathcal{O}_\mathfrak{p}}$ is also generated by them, because of Lemma 3.1. In other words, for any $f \in M_k(\Gamma_2)_{\mathcal{O}_\mathfrak{p}}$ (k: even), there exists an isobaric polynomial $P \in \mathcal{O}_\mathfrak{p}[x_1, \cdots, x_{14}]$ such that $f = P(X_4, \cdots, X_{48})$. Therefore, a similar argument as in the proof of Theorem 3 of Nagaoka [6] implies that, for any $f \in M_k(\Gamma_2)_{\mathcal{O}_\mathfrak{p}}$, there exists a polynomial $Q \in \mathbf{F}[x_1, x_2, x_3]$ such that $\widetilde{f} = Q(X_{10}, Y_{12}, X_{16})$. Here, X_{10}, Y_{12}, X_{16} are algebraically independent over $\mathbf{F}$ and hence the polynomial Q is uniquely determined.

Recall that

$$X_{10} = (q_{12}^{-1} - 2 + q_{12}) q_1 q_2 + \cdots,$$
$$Y_{12} = q_1 + q_2 + \cdots,$$
$$X_{16} = q_1 q_2 + \cdots.$$

Hence, we can write f as

$$\text{(3.1)} \quad \widetilde{f} = \sum_{a,b,c} \gamma_{abc} X_{10}^a Y_{12}^b X_{16}^c \quad \text{(finite sum)}$$
$$= \sum_{a,b,c} (\gamma_{abc} q_{12}^{-a} q_1^{a+b+c} q_2^{a+c} \cdots), \quad (\gamma_{abc} \in \mathbf{F}).$$

Now we consider a sufficient range of a, b, c in the summation of (3.1). Again by Nagaoka's calculation in [6], we have

$$X_4 \equiv X_6 \equiv 1 \bmod \mathfrak{p}, \quad X_{12} \equiv X_{10} \bmod \mathfrak{p},$$
$$X_{18} \equiv X_{16} \bmod \mathfrak{p}, \quad X_{24} \equiv X_{10} X_{16} \bmod \mathfrak{p},$$
$$X_{28} \equiv X_{30} \equiv X_{16}^2 \bmod \mathfrak{p}, \quad X_{36} \equiv X_{10} X_{16}^2 \bmod \mathfrak{p},$$
$$X_{40} \equiv X_{42} \equiv X_{16}^3 \bmod \mathfrak{p},$$
$$X_{48} \equiv X_{16}^4 + X_{10} X_{16}^3 + X_{10}^4 Y_{12} \bmod \mathfrak{p},$$

when $\mathfrak{p} \mid 2$, and also

$$X_4 \equiv X_6 \equiv 1 \bmod \mathfrak{p}, \quad X_{12} \equiv X_{10} \bmod \mathfrak{p},$$
$$X_{18} \equiv X_{16} \bmod \mathfrak{p}, \quad X_{24} \equiv X_{10} X_{16} \bmod \mathfrak{p},$$
$$X_{28} \equiv X_{30} \equiv X_{16}^2 \bmod \mathfrak{p},$$
$$X_{36} \equiv X_{16}^3 + 2X_{10}^3 Y_{12} + X_{10} X_{16}^2 \bmod \mathfrak{p},$$
$$X_{40} \equiv X_{16}^3 + 2X_{10}^3 Y_{12} \bmod \mathfrak{p},$$
$$X_{42} \equiv X_{10}^3 Y_{12} + X_{16}^3 \bmod \mathfrak{p},$$
$$X_{48} \equiv X_{10} X_{16}^3 + 2X_{10}^4 Y_{12} \bmod \mathfrak{p},$$

when $\mathfrak{p} \mid 3$. Assume that $X_k = Q_k(X_{10}, Y_{12}, X_{16})$ $(k = 10, \cdots, 48)$ with $Q_k \in \mathcal{O}_\mathfrak{p}[x_1, x_2, x_3]$. These congruences imply that $\deg Q_k(x, x, x) \le 5k/48$ (total degree) and $\deg Q_k(x, 1, x) \le k/10$ for all k with $k = 4, \cdots, 48$. Since any $f \in M_k(\Gamma_2)_{\mathcal{O}_\mathfrak{p}}$ (k: even) can be written as a polynomial of the fourteen generators X_k $(k = 10, \cdots, 48)$ and Y_{12}, for f with the description

$$\widetilde{f} = \sum_{a,b,c} \gamma_{abc} X_{10}^a Y_{12}^b X_{16}^c,$$

it suffices to move a, b, c in a range $a + b + c \le 5k/48$, $a + c \le k/10$. Namely we can write as

$$\text{(3.2)} \qquad \widetilde{f} = \sum_{\substack{0 \le a,b,c \\ a+b+c \le 5k/48 \\ a+c \le k/10}} \gamma_{abc} X_{10}^a Y_{12}^b X_{16}^c.$$

Let a_0, b_0, c_0 be the minimum numbers of a, b, c appearing the summation of (3.2), respectively. Then we have $\gamma_{a_0b_0c_0} \equiv a_f(a_0+b_0+c_0, -a_0, a_0+c_0)$ mod $\mathfrak{p}$, since

$$(a,b,c) \neq (a',b',c') \quad \text{implies} \quad q_{12}^{-a} q_1^{a+b+c} q_2^{a+b} \neq q_{12}^{-a'} q_1^{a'+b'+c'} q_2^{a'+b'}.$$

Finally, we suppose that f satisfies $a_f(m,r,n) \equiv 0 \bmod \mathfrak{p}$ for all m, $n \leq k/10$ and r with $4mn - r^2 \geq 0$. By the original Sturm bounds for elliptic modular forms, we have $\Phi(f) \equiv 0 \bmod \mathfrak{p}$, where Φ is the Siegel Φ-operator. From this fact, it is easy to see that $\widetilde{f}$ is a multiple of X_{10} or X_{16}.

On the other hand, by an inductive argument, we have $\gamma_{abc} \equiv a_f(a+b+c, -a, a+c) \equiv 0 \bmod \mathfrak{p}$ for all a, c with $a+c \leq k/10$. Then we get a description

$$\widetilde{f} = \sum_{k/10 < a+c \leq k/10} \gamma_{abc} X_{10}^a Y_{12}^b X_{16}^c.$$

However, the summation above is empty. Therefore we obtain $\widetilde{f} = 0$. This completes the proof of (1). □

Proof of (2). Let k be odd and $f \in M_k(\Gamma)_{\mathcal{O}_\mathfrak{p}}$. Then $fX_{35} \in M_{k+35}(\Gamma)_{\mathcal{O}_\mathfrak{p}}$ is of even weight. If we apply (1) to fX_{35}, we obtain the assertion of (2). □

3.2. Proof for the case of level *N*. For the case of level N, we can reduce it to the case of level 1, by taking a norm of modular forms as in [2].

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