Some problems of hypergeometric integrals associated with hypersphere arrangement

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Abstract: The n dimensional hypergeometric integrals associated with a hypersphere arrangement S are formulated by the pairing of n dimensional twisted cohomology $H^n_\nabla(X, \Omega^\cdot(*S))$ and its dual. Under the condition of general position there are stated some results and conjectures which concern a representation of the standard form by a special basis of the twisted cohomology, the variational formula of the corresponding integral in terms of special invariant 1-forms using Calyley-Menger minor determinants, a connection relation of the unique twisted n-cycle identified with the unbounded chamber to a special basis of twisted n-cycles identified with bounded chambers. General conjectures are presented under a much weaker assumption.

Key words: Hypergeometric integral; hypersphere arrangement; twisted rational de Rham cohomology; Cayley-Menger determinant; contiguity relation; Gauss-Manin connection.

1. Preliminary. Hypersphere arrangements are an interesting subject in analysis and geometry for a long time (see [16] for example). The purpose of this note is to present some problems and results in relation to hypergeometric integrals. The details in case where the dimension $n \le 3$, the number $m = n+1$ of hyperspheres will be presented in a forthcoming paper.

Let $\mathcal{A}$ be an arrangement of $n-1$ dimensional hyperspheres in the complex n dimensional affine space $\mathbf{C}^n$:

$$S_j : f_j(x) = Q(x) + 2(\alpha_j, x) + \alpha_{j0} = 0 \quad (1 \le j \le m),$$

where

$$Q(x) = \sum_{\nu=1}^n x_\nu^2,\ (\alpha_j, x) = \sum_{\nu=1}^n \alpha_{j\nu} x_\nu,$$
$$\alpha_j = (\alpha_{j1}, \dots, \alpha_{jn}) \in \mathbf{R}^n,\ \alpha_{j0} \in \mathbf{R}.$$

S_j represents the $n-1$ dimensional (complex) hypersphere with center $O_j = -\alpha_j$ and with radius r_j such that $r_j^2 = -\alpha_{j0} + Q(\alpha_j)$. The distance ρ_{ij} between O_i and O_j is given by $\rho_{ij}^2 = Q(\alpha_i - \alpha_j)$.

Let X be the complement of the union $S = \bigcup_{j=1}^m S_j$ in $\mathbf{C}^n$. Denote by $\Omega^\cdot(X, *S) = \oplus_{p=0}^n \Omega^p(X, *S)$ the space of rational differential forms on $\mathbf{C}^n$ which are holomorphic in X.

Let $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbf{C}^m$ be a system of m tuple of exponents such that

$$\Phi(x) = \prod_{j=1}^m f_j(x)^{\lambda_j} \quad (\lambda_j \in \mathbf{C})$$

defines a multiplicative meromorphic function on $\mathbf{C}^n$. The covariant differentiation associated with $\Phi(x)$ is defined as follows:

$$\nabla\psi = d\psi + d\log\Phi \wedge \psi \quad (\psi \in \Omega^\cdot(X, *S)).$$

$H^*_\nabla(X, \Omega^\cdot(*S))$ denotes the corresponding rational de Rham cohomology. $\mathcal{L}$ and $\mathcal{L}^*$ denote the local system and its dual on X attached to $\Phi(x)$.

Let ϖ be the standard n-form

$$\varpi = dx_1 \wedge \cdots \wedge dx_n.$$

Take a twisted cycle $\mathfrak{z} \in H_n(X, \mathcal{L}^*)$ and consider the integral of $\varphi\varpi \in H^n_\nabla(X, \Omega^n(*S))$,

$$\langle \varphi, \mathfrak{z} \rangle = \int_{\mathfrak{z}} \Phi(x)\varphi\varpi,$$

which defines the perfect pairing between $H^n_\nabla(X, \Omega^n(*S))$ and $H_n(X, \mathcal{L}^*)$. This fact is due to A. Grothendieck and P. Deligne (see [10]).

Differential and difference structures related to $\langle \varphi, \mathfrak{z} \rangle$ can be described in terms of invariants with respect to the isometry group for the arrangement of hyperspheres (see [3–6,9] for general treatment).

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Notation. Denote by ε_j $(1 \le j \le m)$ the standard basis of $\mathbf{C}^m$ so that $\lambda = \sum_{j=1}^m \lambda_j \varepsilon_j$.

Denote by $[1,m]$ the set of indices $1,2,\dots,m$. For $J = \{j_1,\dots,j_p\} \subset [1,m]$, we denote by $|J| = p$ the size of J, by $\partial_\nu J$ $(1 \le \nu \le p)$ the subset $\{j_1,\dots,j_{\nu-1},j_{\nu+1},\dots,j_p\}$. $I^c = [1,m] - I$ denotes the complement of I in $[1,m]$. We say $J \subset [1,m]$ to be "admissible" if $1 \le |J| \le n+1$. The family of all admissible sets is denoted by $\mathcal{B}$.

Definition 1. Let $B = (b_{ij})_{1\le i,j \le m+2}$ be the symmetric matrix of degree $m+2$ whose components of the i th row and the j th column are

$$b_{jj} = 0,\ b_{1j} = 1\ (2 \le j \le m+2),$$
$$b_{2j} = r_{j-2}^2\ (3 \le j \le m+2),$$
$$b_{ij} = \rho_{i-2\,j-2}^2\ (3 \le i < j \le m+2).$$

This is called a Cayley-Menger matrix associated with the arrangement $\mathcal{A}$. Cayley-Menger determinants are defined to be minors including the first row and the first column (see [11,12]). Namely for $I = \{i_1,i_2,\dots,i_p\}, J = \{j_1,j_2,\dots,j_p\} \subset [1,m]$,

$$B\begin{pmatrix} 0 & I \\ 0 & J \end{pmatrix} = B\begin{pmatrix} 0 & i_1 & \cdots i_p \\ 0 & j_1 & \cdots j_p \end{pmatrix} = \begin{vmatrix} 0 & 1 & \dots & 1 \\ 1 & \rho_{i_1j_1}^2 & \dots & \rho_{i_1j_p}^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \rho_{i_pj_1}^2 & \dots & \rho_{i_pj_p}^2 \end{vmatrix},$$

$$B\begin{pmatrix} 0 & \star & \partial_1 I \\ 0 & j_1 & \partial_1 J \end{pmatrix} = B\begin{pmatrix} 0 & \star & i_2 & \cdots i_p \\ 0 & j_1 & j_2 & \cdots j_p \end{pmatrix} = \begin{vmatrix} 0 & 1 & \dots & 1 \\ 1 & r_{j_1}^2 & \dots & r_{j_p}^2 \\ 1 & \rho_{i_2j_1}^2 & \dots & \rho_{i_2j_p}^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \rho_{i_pj_1}^2 & \dots & \rho_{i_pj_p}^2 \end{vmatrix},$$

$$B\begin{pmatrix} 0 & \star & \partial_1 I \\ 0 & \star & \partial_1 J \end{pmatrix} = B\begin{pmatrix} 0 & \star & i_2 & \cdots i_p \\ 0 & \star & j_2 & \cdots j_p \end{pmatrix} = \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & r_{j_2}^2 & \dots & r_{j_p}^2 \\ 1 & r_{i_2}^2 & \rho_{i_2j_2}^2 & \dots & \rho_{i_2j_p}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & r_{i_p}^2 & \rho_{i_pj_2}^2 & \dots & \rho_{i_pj_p}^2 \end{vmatrix}.$$

$B\begin{pmatrix} 0 & I \\ 0 & J \end{pmatrix}$ will be abbreviated by $B(0\,I)$ if $I = J$, in the same way.

$B\begin{pmatrix} 0 & \star & \partial_1 I \\ 0 & \star & \partial_1 J \end{pmatrix}$ will be abbreviated by $B(0 \star \partial_1 I)$ if $\partial_1 I = \partial_1 J$.

For example we have

$$B\begin{pmatrix} 0 & i & j \\ 0 & k & l \end{pmatrix} = \rho_{il}^2 + \rho_{jk}^2 - \rho_{ik}^2 - \rho_{jl}^2,$$
$$B\begin{pmatrix} 0 & \star & j \\ 0 & k & l \end{pmatrix} = r_l^2 + \rho_{jk}^2 - r_k^2 - \rho_{jl}^2,$$
$$B\begin{pmatrix} 0 & \star & j \\ 0 & \star & l \end{pmatrix} = r_j^2 + r_l^2 - \rho_{jl}^2,$$
$$B(0ij) = 2\rho_{ij}^2, B(0\star j) = 2r_j^2.$$

We impose the following two conditions

$(\mathcal{H}1)$: (i) $(-1)^p B(0\,I) > 0$

(for any admissible I, $1 \le p \le n+1$),

(ii) $(-1)^{p-1} B(0 \star I) > 0$

(for any admissible I, $1 \le p \le n+1$),

where $I = \{i_1,\dots,i_p\}$.

The singularity defined by the equations $B(0\,I) = 0$ or $B(0 \star I) = 0$ is nothing else than Landau singularity associated with the integral $\langle \varphi, \mathfrak{z} \rangle$ (see [15]).

$(\mathcal{H}2)$: $\underline{\lambda_j \text{ are all positive.}}$

Lemma 2. *Suppose that λ satisfies the conditions, for $J = \{j_1,\dots,j_r\} \subset [1,m]$,*

$$\lambda_{j_1} + \cdots + \lambda_{j_r} \notin \mathbf{Z}, \quad (1 \le r \le n),$$
$$-2\lambda_\infty + \lambda_{j_1} + \cdots + \lambda_{j_r} \notin \mathbf{Z}, \quad (0 \le r \le n-1).$$

Then the following fact holds:

(i) $H_\nabla^p(X, \Omega^\cdot(*S)) \cong \{0\} \quad (0 \le p \le n-1)$,

(ii) $\dim H_\nabla^n(X, \Omega^\cdot(*S)) = |Eu(X)|$

$$= \sum_{\nu=1}^n \binom{m}{\nu} + \binom{m-1}{n},$$

where $Eu(X)$ represents the Euler *number of X.*

For the proof see [1,7,8].

2. Statement of problems. From now on, we assume that $m = n+1$.

Denote by $K_j : \mathbf{R}^n \cap \{f_j(x) \le 0\}$ the closure of the inside of the real part $\Re S_j = S_j \cap \mathbf{R}^n$ in $\mathbf{R}^n$.

Under the condition $(\mathcal{H}1)$, the number of bounded connected components of $\mathbf{R}^n - \bigcup_{j=1}^m S_j$ is equal to $|Eu(X)| = 2^{n+1} - 1$. It is also equal to $\dim H_n(X, \mathcal{L}^*)$. The twisted cycles corresponding to these bounded chambers constitute a basis of $H_n(X, \mathcal{L}^*)$.

More precisely,

Lemma 3. *For every admissible set I with $|I| = n$, the intersection $\bigcap_{i\in I} S_i$ consists of two different points. Moreover for every admissible $I \in \mathcal{B}$ we see*

$$K_I = \text{the closure of } \left\{ \bigcap_{i\in I} K_i - \bigcup_{j\in I^c} K_j \right\} \neq \emptyset$$

has an inner point. Each K_I can be identified with a twisted cycle $\mathfrak{z}_I$ representing a homology class in $H_n(X, \mathcal{L}^*)$. The twisted cycles $\mathfrak{z}_I$ $(I \in \mathcal{B})$ form a basis of $H_n(X, \mathcal{L}^*)$.

For the proof see [7,8].

On the other hand,

Lemma 4. *$H^n_\nabla(X, \Omega^{\cdot}(*S))$ is spanned by*

$$F_I := \frac{\varpi}{f_{i_1} \cdots f_{i_p}} \quad (1 \leq p \leq n+1) \quad (I \in \mathcal{B}),$$

or equivalently by

$$W_0(I)\varpi := -\sum_{\nu=1}^{p} B\begin{pmatrix} 0 & \star & \partial_\nu I \\ 0 & i_\nu & \partial_\nu I \end{pmatrix} F_{\partial_\nu I} + B(0 \star I) F_I \quad (I \in \mathcal{B}).$$

$(\mathcal{H}1)$ assures that $\{F_I\,(I \in \mathcal{B})\}$ or $\{W_0(I)\varpi\,(I \in \mathcal{B})\}$ constitutes a basis of $H^n_\nabla(X, \Omega^{\cdot}(*S))$. The former will be called "of first kind" and the latter will be called "of second kind". Both are related to each other by a triangular matrix. See also [7,8].

Using the basis of the second kind, we give the following conjecture.

Conjecture I. ϖ is represented cohomologically in terms of the basis of second kind

$$(1) \quad (2\lambda_\infty + n)\, \varpi \sim \sum_{p=1}^{n+1} \sum_{I\in\mathcal{B}, |I|=p} (-1)^p \frac{\prod_{j\in I} \lambda_j}{\prod_{\nu=1}^{p-1} (\lambda_\infty + n - \nu)} W_0(I)\varpi$$

in $H^n_\nabla(X, \Omega^{\cdot}(*S))$ ($\sim$ means "cohomologous").

$\langle \varphi, \mathfrak{z} \rangle$ is an analytic function of the parameters $\alpha_{j\nu}$. The total differentiation d_B of $\langle \varphi, \mathfrak{z} \rangle$ with respect to the parameters $\alpha_{j\nu}$ has the expression

$$d_B \langle \varphi, \mathfrak{z} \rangle = \int_{\mathfrak{z}} \Phi(x) \nabla_B(\varphi\varpi),$$

where

$$(2) \qquad \nabla_B(\varphi\varpi) = (d_B\varphi + d_B \log \Phi\, \varphi)\varpi.$$

In order to express the RHS of (2), we introduce the following differential 1-forms θ_J $(J \in \mathcal{B})$:

Definition 5.

$$\theta_j = -\frac{1}{2} d\log(r_j^2),$$

$$\theta_{jk} = \frac{1}{2} d\log \rho_{jk}^2,$$

$$\theta_{jkl} = -\frac{1}{2} \left\{ \frac{B\begin{pmatrix} 0 & j & k & l \\ 0 & \star & k & l \end{pmatrix}}{B(0jkl)} d\log \rho_{kl}^2 + \frac{B\begin{pmatrix} 0 & k & j & l \\ 0 & \star & j & l \end{pmatrix}}{B(0jkl)} d\log \rho_{jl}^2 + \frac{B\begin{pmatrix} 0 & l & j & k \\ 0 & \star & j & k \end{pmatrix}}{B(0jkl)} d\log \rho_{jk}^2 \right\}.$$

More generally for $J = \{j_1, \ldots, j_p\} \in \mathcal{B}$ $(2 \leq p \leq n+1)$,

$$\theta_J := \frac{(-1)^p}{2} \sum_{\{L\}=\{J\}; l_1<l_2} d\log \rho_{l_1 l_2}^2 \cdot \frac{B\begin{pmatrix} 0 & \star & l_1 & l_2 \\ 0 & l_3 & l_1 & l_2 \end{pmatrix} B\begin{pmatrix} 0 & \star & l_1 & l_2 & l_3 \\ 0 & l_4 & l_1 & l_2 & l_3 \end{pmatrix}}{\prod_{\nu=3}^{p} B(0 l_1 l_2 l_3 \cdots l_\nu)} \cdots B\begin{pmatrix} 0 & \star & l_1 & l_2 & l_3 & \cdots & l_{p-1} \\ 0 & l_p & l_1 & l_2 & l_3 & \cdots & l_{p-1} \end{pmatrix},$$

where $L = \{l_1, l_2, \ldots, l_p\}$ run over the set of sequences such that L coincides with J as a set in $[1, m]$ and satisfies $l_1 < l_2 < l_3 < l_4 < \ldots < l_p$.

The second conjecture can be stated in the following form (Gauss-Manin connection):

Conjecture II.

$$(3) \quad \nabla_B \varpi \sim \sum_{p=1}^{n+1} V_p\, \varpi,$$

$$V_p = \sum_{J\in\mathcal{B}, |J|=p} \frac{\prod_{j\in J} \lambda_j}{\prod_{\nu=1}^{p-1} (\lambda_\infty + n - \nu)} \theta_J W_0(J).$$

It seems remarkable that in the RHS of (3) the expression of θ_J is independent of n and depends only on J for any fixed admissible J.

Finally we state a conjecture concerning the connection formula among twisted cycles.

For $J = \{j_1, \dots, j_p\} \subset [1, m]$ $(1 \le p \le m)$, $\mathfrak{z}_J$ $(J \in \mathcal{B})$ forms a basis $H_n(X, \mathcal{L}^*)$. The complement $K^{[1,m]} = \mathbf{R}^n - \bigcup_{j \in [1,m]} K_j$ can also be regarded as a twisted n-cycle denoted by $\mathfrak{z}_\infty$. We put further $J^c = [1, m] - J$, $\lambda_J = \sum_{j \in J} \lambda_j$, (In case $J = \emptyset$, we put $\lambda_J = 1$), $\lambda_\infty = \sum_{j \in [1,m]} \lambda_j$.

We can now state:

Conjecture III. The following connection formula holds ($\sim$ means "homologous"):

(i) Case where n even,

$$\mathfrak{z}_\infty \sim - \sum_{J \in \mathcal{B}, n \ge |J|} \frac{\sin \pi \lambda_{J^c}}{\sin \pi \lambda_\infty} \mathfrak{z}_J.$$

(ii) Case where n odd,

$$\mathfrak{z}_\infty \sim - \sum_{J \in \mathcal{B}} \frac{\cos \pi \lambda_{J^c}}{\cos \pi \lambda_\infty} \mathfrak{z}_J.$$

For example, in case $n = 1$,

$$\mathfrak{z}_\infty \sim - \frac{1}{\cos \lambda_\infty} \mathfrak{z}_{12} - \frac{\cos \lambda_2}{\cos \lambda_\infty} \mathfrak{z}_1 - \frac{\cos \lambda_1}{\cos \lambda_\infty} \mathfrak{z}_2.$$

In case $n = 2$,

$$\begin{aligned} \mathfrak{z}_\infty \sim & - \frac{\sin \pi \lambda_1}{\sin \pi \lambda_\infty} \mathfrak{z}_{23} - \frac{\sin \pi \lambda_2}{\sin \pi \lambda_\infty} \mathfrak{z}_{13} - \frac{\sin \pi \lambda_3}{\sin \pi \lambda_\infty} \mathfrak{z}_{12} \\ & - \frac{\sin \pi (\lambda_2 + \lambda_3)}{\sin \pi \lambda_\infty} \mathfrak{z}_1 - \frac{\sin \pi (\lambda_1 + \lambda_3)}{\sin \pi \lambda_\infty} \mathfrak{z}_2 \\ & - \frac{\sin \pi (\lambda_1 + \lambda_2)}{\sin \pi \lambda_\infty} \mathfrak{z}_3. \end{aligned}$$

We can prove the following

Theorem 6. *In case where $n = 1, 2, 3$, Conjectures I, II, and III affirmatively hold.*

The proof can be done by using contiguity relations involved in $\langle \varphi, \mathfrak{z} \rangle$ relative to the shifts $\lambda \to \lambda \pm \varepsilon_j$.

The formula (3) can be regarded as an extension of the classical variation formula due to L. Schläfli concerning the volume of a geodesic simplex in the unit hypersphere (see [2,17,18]). In fact, by taking the limit of (3) for $\lambda \to 0$, we can derive the variation formula of the volume of a real domain bounded by hyperspheres.

3. Generalization. In this section we assume m $(m \ge n + 2)$ is arbitrary. Denote by e_J $(J = \{j_1, \dots, j_p\} \subset [1, m],\ p \le n)$ the logarithmic p-form $d \log f_{i_1} \wedge \cdots \wedge d \log f_{i_p}$.

Fix an arbitrary subset $J = \{j_1, j_2, \dots, j_{n+1}\} \subset [1, m]$. Then under the condition $(\mathcal{H}1)$, we have

$$(4) \quad \sum_{\nu=1}^{n+1} (-1)^{\nu-1} e_{\partial_\nu J} = \frac{2^{\frac{n}{2}}}{\sqrt{(-1)^{n+1} B(0\, J)}} W_0(J) \varpi.$$

Fix a subset $I = \{i_1, i_2, \dots, i_{n+2}\} \subset [1, m]$. Then as a consequence of (4) the following fundamental equality holds among F_J $(J \in \mathcal{B})$:

$$(5) \qquad \sum_{\nu=1}^{n+2} \pm \frac{W_0(\partial_\nu I) \varpi}{\sqrt{(-1)^{n+1} B(0\, \partial_\nu I)}} = 0.$$

Moreover the following partial fraction decomposition holds (note that $B(0\, I) = 0$, $(-1)^{n+1} B(0\, \partial_\nu I) > 0$ and $(-1)^n B(0 \star I) \ge 0$):

$$(6) \qquad F_I = \sum_{\nu=1}^{n+2} \pm \left(- \frac{B(0\, \partial_\nu I)}{B(0 \star I)} \right)^{1/2} F_{\partial_\nu I},$$

so that F_I can be expressed as a linear combination of F_J $(J \in \mathcal{B})$ provided $B(0 \star I) \ne 0$. Here the signs $\pm$ in the RHS of (5), (6) can be taken such that the equalities hold

$$(7) \quad \pm \sqrt{B(0\, \partial_\mu I)\, B(0\, \partial_\nu I)} = B \begin{pmatrix} 0 & i_\mu & \partial_\mu \partial_\nu I \\ 0 & i_\nu & \partial_\mu \partial_\nu I \end{pmatrix}.$$

Note that owing to Jacobi identity and the above assumption the square of the LHS equals the square of the RHS in (7).

For $Q(x) = \sum_{\nu=1}^n x_\nu^2$, let

$$\begin{aligned} *dQ = & \sum_{\nu=1}^{n} (-1)^{\nu-1} x_\nu dx_1 \wedge \cdots \wedge dx_{\nu-1} \\ & \wedge dx_{\nu+1} \wedge \cdots \wedge dx_n. \end{aligned}$$

In addition to the above identities, there are cohomologous relations like

$$(8) \quad \nabla(e_J) \sim 0, \quad |J| = n - 1,$$

$$(9) \quad \nabla\left(\frac{*dQ}{f_{j_1} \cdots f_{j_r}} \right) \sim 0,$$

$$J = \{j_1, \dots, j_r\} \subset [1, m], \quad 0 \le r \le n + 1.$$

These identities (4)–(9) seem sufficient to prove the above Conjectures I, II, and III.

In view of the results obtained in [13,14] in case of hyperplane arrangement, it seems natural to make the following conjecture in case of hypersphere arrangement.

Conjecture IV. Let $\mathcal{A} = \{S_1, \dots, S_m\}$ be an arbitrary arrangement of hyperspheres i.e., α_j, α_{j0} be arbitrary.

In addition to $(\mathcal{H}2)$, assume further that

$(\mathcal{H}3)$: For any choice of $I \subset [1, m]$ such that $|I| \le n$, $\bigcap_{j \in I} \Re S_j \ne \emptyset$.

Then

(i) If λ is generic, $H^n_\nabla(X, \Omega^\cdot(*S))$ is spanned by F_I $(I \in \mathcal{B})$. However these are no more necessarily linearly independent. Under the condition $(\mathcal{H}1)$, (5) are the fundamental relations satisfied by them.

(ii) $|Eu(X)|$ which equals $\dim H_n(X, \mathcal{L}^*)$ also equals the number of bounded connected chambers of $\mathbf{R}^n - S$.

Remark 7. It seems interesting to extend the above formulae stated in Conjectures I and II to arbitrary m by using the differential forms F_I or $W_0(I)\varpi$ $(I \in \mathcal{B})$ under $(\mathcal{H}1)$ or even without $(\mathcal{H}1)$.

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