Auxiliary differential polynomials for the first Painlevé hierarchy

By Yoshikatsu SASAKI

Department of Mathematics, Hiroshima University,
1-3-1 Kagamiyama, Higashi-Hiroshima, Hiroshima 739-8526, Japan

Abstract: This article concerns with the first Painlevé hierarchy, i.e. the $2n$-th order analogues of the first Painlevé equation. Several higher order analogues of the first Painlevé equation are proposed by several authors however, we investigate one derived from the KdV hierarchy by studying singularity manifold. We give the auxiliary differential polynomials of the first Painlevé hierarchy.

Key words: Painlevé hierarchy; differential polynomials.

1. Introduction.

1.1. The first Painlevé hierarchy. Let D or $'$ stand for the differentiation with respect to z, and D^{-1} stand for the inverse operator of D.

Consider the serial equations

$$(_{2n}P_{\mathrm{I}}) \qquad d_n[w] + 4z = 0$$

for $n \in \mathbf{N}$, where $d_n[w]$ is an expression of a given meromorphic function w defined by $d_0[w] = -4w$ and $Dd_{n+1}[w] = (D^3 - 8wD - 4w')d_n[w]$. The equations are derived from the singular maniford equation for the KdV hierarchy, and we call them *the first Painlevé hierarchy* [4,5,9]. For example, if $n = 1$, $d_1/4 = -w'' + 6w^2$, then $(_2P_{\mathrm{I}})$ coincides the first Painlevé equation

$$(P_{\mathrm{I}}) \qquad w'' = 6w^2 + z.$$

If $n = 2$, $d_2/4 = -w^{(4)} + 20ww'' + 10w'^2 - 40w^3$, then we have

$$(_4P_{\mathrm{I}}) \qquad w^{(4)} = 20ww'' + 10w'^2 - 40w^3 + z.$$

Shimomura [8] proved that each $d_n[w]$ is a differential polynomial of $2n$-th order, i.e. each $(_{2n}P_{\mathrm{I}})$ is an ordinary differential equation of $2n$-th order.

On the other hand, *the second Painlevé hierarchy* [1–5]

$$(_{2n}P_{\mathrm{II}}) \qquad D^{-1}S_w^n Dw + zw + \alpha = 0$$

for $n \in \mathbf{N}$ with a complex parameter α, defined by $S_w := -D^2 + 4w'D^{-1}w + 4w^2$, are obtained by sim-

ilarity reduction from *the KdV hierarchy* [6]. For example, if $n = 1$, $D^{-1}S_w Dw = -w'' + 2w^3$, then $(_2P_{\mathrm{II}})$ coincides the second Painlevé equation

$$(P_{\mathrm{II}}) \qquad w'' = 2w^3 + zw + \alpha.$$

The author proved [7] that each $\tilde{d}_n[w] := D^{-1}S_w^n Dw$ is a differential polynomial of $2n$-th order, i.e. $(_{2n}P_{\mathrm{II}})$ is an ordinary differential equation of $2n$-th order. In the proof of the above, auxiliary differential polynomials $\tilde{c}_n[w] := D^{-1}wD\tilde{d}_n[w]$ were introduced, and played important roles.

1.2. Result. It is natural to guess that the first Painlevé hierarchy $(_{2n}P_{\mathrm{I}})$ also has the auxiliary differential polynomials which gives more informations in the study of $(_{2n}P_{\mathrm{I}})$. Now we define an expression c_n. Our main theorem is as follows:

Theorem A. *Define $c_n[w]$ by $Dc_n[w] = wDd_n[w]$. Then each c_n is a differential polynomial of $2n$-th order.*

2. Proof of the theorem.

Now, we recall the definition of $d_n[w]$: $d_0[w] = -4w$ and $Dd_{n+1}[w] = (D^3 - 8wD - 4w')d_n[w]$. Note that $(_{2n}P_{\mathrm{I}})$ is written as $d_n[w]/4 + z = 0$. We make the constant of integration in $d_n[w]$ vanish as in the book of GLS [4]. Recall the first 3 entries:

$$d_0[w]/4 = -w,$$
$$d_1[w]/4 = -w'' + 6w^2,$$
$$d_2[w]/4 = -w^{(4)} + 20ww'' + 10w'^2 - 40w^3.$$

We also recall that we introduced $c_n[w]$ by $Dc_n[w] = wDd_n[w]$. Again, we make the constant of integration in $c_n[w]$ vanish as in the definition of $d_n[w]$. For example,

$$c_0[w]/4 = D^{-1}wDd_0[w]/4 = -D^{-1}wDw = -\frac{1}{2}w^2,$$

$$c_1[w]/4 = D^{-1}wDd_1[w]/4 = -ww'' + \frac{1}{2}w'^2 + 4w^3,$$

$$c_2[w]/4 = D^{-1}wDd_2[w]/4 = -ww^{(4)} + w'w^{(3)} - \frac{1}{2}w''^2$$
$$+ 20w^2w'' - 30w^4.$$

Proof of Theorem A. By induction. As is shown above, $d_n[w]$ and $c_n[w]$ are differential polynomials for $n = 0, 1, 2$. Assume that $d_k[w]$ and $c_k[w]$ are differential polynomials for k less than or equal to n. By definition of $d_n[w]$ and $c_n[w]$, using integration by parts, we have

$$Dd_{n+1}[w] = (D^3 - 8wD - 4w')d_n[w]$$
$$= D^3 d_n[w] - 8wDd_n[w] - 4w'd_n[w]$$
$$= D(D^2 d_n[w] - 4c_n[w] - 4wd_n[w]),$$

which implies that $d_{n+1}[w]$ is a differential polynomial. Multiplying both sides of $Dd_{k+1}[w] = D(D^2 d_k[w] - 4c_k[w] - 4wd_k[w])$ by $d_{n-k}[w]$ and summing-up for $k = 0, 1, \ldots, n$, we have

$$(1) \quad \sum_{k=0}^{n} d_{n-k}[w]Dd_{k+1}[w]$$
$$= \sum_{k=0}^{n} d_{n-k}[w]D^3 d_k[w]$$
$$- 4\sum_{k=0}^{n} d_{n-k}[w](Dc_k[w] + Dwd_k[w]).$$

The l.h.s. of (1) is written as

$$\sum_{k=0}^{n} d_{n-k}[w]Dd_{k+1}[w]$$
$$= -4wDd_{n+1}[w] + \sum_{k=0}^{n-1} d_{n-k}[w]Dd_{k+1}[w]$$
$$= -4Dc_{n+1}[w] + \frac{1}{2}D\sum_{k=0}^{n-1} d_{n-k}[w]d_{k+1}[w].$$

The 1st term of the r.h.s. of (1) is written as

$$\sum_{k=0}^{n} d_{n-k}[w]D^3 d_k[w]$$
$$= D\left[\sum_{k=0}^{n} d_{n-k}[w]D^2 d_k[w] - \frac{1}{2}\sum_{k=0}^{n} Dd_{n-k}[w]Dd_k[w]\right].$$

The 2nd term of the r.h.s. of (1) is written as

$$-4\sum_{k=0}^{n} d_{n-k}[w]D(c_k[w] + wd_k[w])$$
$$= -4D\sum_{k=0}^{n} wd_{n-k}[w]d_k[w].$$

Therefore, we obtain

$$4Dc_{n+1}[w]$$
$$= D\left[4\sum_{k=0}^{n} wd_{n-k}[w]d_k[w] + \frac{1}{2}\sum_{k=0}^{n-1} d_{n-k}[w]d_{k+1}[w]\right.$$
$$\left. - \sum_{k=0}^{n} d_{n-k}[w]D^2 d_k[w] + \frac{1}{2}\sum_{k=0}^{n} Dd_{n-k}[w]Dd_k[w]\right]$$

which implies that $c_{n+1}[w]$ is also a differential polynomial. Recall the fact proved by Shimomura [8] that $d_n[w] = -w^{(2n)} + [(2n - 2)$-th order term]. The order of $c_n[w]$ is determined as the statement of the theorem immediately by the above fact and the definition of $c_n[w]$. $\square$

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