Middle convolution and non-Schlesinger deformations

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Abstract: Middle convolution is an operation for Fuchsian systems of differential equations which preserves Schlesinger's deformation equations. In this paper we announce that Bolibruch's non-Schlesinger deformations of Fuchsian systems are, in general, not preserved by middle convolution.

Key words: Middle convolution; isomonodromic deformations; non-Schlesinger isomonodromic deformations.

1. Introduction. Various aspects of middle convolution have recently attracted a lot of attention (see, for example, [7–9, 12–14]). Middle convolution is an operation for non-resonant Fuchsian systems of differential equations which preserves deformation equations [9]. In particular, the so-called Hitchin systems [10] obtained from the Schlesinger systems (see below) are invariant under middle convolution. It is known that there exist non-Schlesinger deformations for resonant Fuchsian systems and it is natural to expect that they are also preserved by middle convolution. In this paper we announce that in general non-Schlesinger deformations are not preserved by middle convolution. Moreover, the algorithm of middle convolution gives new explicit examples of non-Schlesinger isomonodromic deformations of the resonant Fuchsian systems of order higher than two and which are different from Bolibruch's example for a Fuchsian system of order 2. This paper announces the main results and the details (including explicit examples) will be published separately [2].

2. Isomonodromic deformations. Let us consider a system of p linear differential equations on the Riemann sphere

$$(1)\quad \frac{dy}{dz} = \left(\sum_{i=1}^{n} \frac{A_i^0}{z - a_i^0}\right) y, \quad \sum_{i=1}^{n} A_i^0 = -A_{n+1}^0$$

with singularities $a_1^0, \dots, a_n^0 \in \mathbf{C}$, $a_{n+1}^0 = \infty$. Here $y(z) \in \mathbf{C}^p$. System (1) is the Fuchsian system. One can define its monodromy representation [11] by

$$(2)\quad \chi^0 : \pi_1(T^0, z_0) \to \mathbf{GL}(p, \mathbf{C}), \quad T^0 = \bar{\mathbf{C}} \setminus \cup_{i=1}^{n+1} \{a_i^0\}.$$

System (1) can be included in the isomonodromic family of Fuchsian systems

$$(3)\quad \frac{dy}{dz} = \left(\sum_{i=1}^{n} \frac{A_i(a)}{z - a_i}\right) y, \quad \sum_{i=1}^{n} A_i(a) = -A_{n+1}(a),$$

with $A_i(a^0) = A_i^0$, $a^0 = (a_1^0, \dots, a_n^0)$. The parameters $a = (a_1, \dots, a_n)$ of the family (3) are the locations of singular points. Similarly we can define a monodromy representation of system (3) for $a \in D(a^0) \setminus \cup_{i,j=1, i\neq j}^{n} \{a_i = a_j\}$ by

$$(4)\quad \chi_a : \pi_1(T_a, z_0) \to \mathbf{GL}(p, \mathbf{C}), \quad T_a = \bar{\mathbf{C}} \setminus \cup_{i=1}^{n+1} \{a_i\},$$

where $D(a^0)$ is a small open disk centered at a^0.

Definition 2.1 ([3, 4, 5]). The family of Fuchsian systems (3) is isomonodromic if the monodromy representation χ_a coincides with the representation χ^0 of system (1) for any $a \in D(a^0) \setminus \cup_{i,j=1, i\neq j}^{n} \{a_i = a_j\}$.

The isomonodromuc family (3) is also called the isomonodromic deformation.

The following statement for Definition 2.1 was proved by A. A. Bolibruch.

Theorem 2.1 ([3, 4, 5]). *The family of Fuchsian systems* (3) *is isomonodromic if and only if there exists a matrix-valued differential 1-form* ω *on* $\mathbf{C} \times D(a^0) \setminus \cup_{i=1}^{n} \{z - a_i = 0\}$ *such that*
i) $\omega = \sum_{i=1}^{n} \frac{A_i(a)}{z - a_i} dz$ *for any fixed* $a \in D(a^0)$;
ii) $d\omega = \omega \wedge \omega$.

Definition 2.1 is the general one. The Schlesinger deformations are the most known in the literature. They are given by the differential form

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$$\omega_{Schl} = \sum_{i=1}^{n} \frac{A_i(a)}{z-a_i}\, d(z-a_i). \tag{5}$$

The second condition of Theorem 2.1 is then equivalent to

$$dA_i(a) = -\sum_{j=1, j\neq i}^{n} \frac{[A_i(a), A_j(a)]}{a_i - a_j}\, d(a_i - a_j), \tag{6}$$

which is also known as the Schlesinger equation [11]. The fundamental matrix $Y_{Schl}(z,a)$ of (3) satisfies

$$Y_{Schl}(\infty, a) \equiv C, \tag{7}$$

where C is a constant non-degenerated matrix.

Definition 2.2. Deformations of the Fuchsian systems satisfying (7) are called normalized.

Note that Definition 2.1 does not require any normalization.

An arbitrary isomonodromic deformation is not necessarily the Schlesinger deformation (5). Let us consider a family of Fuchsian systems with the fundamental matrix $Y(z,a) = \Gamma(a)Y_{Schl}(z,a)$, where $\Gamma(a)$ is a holomorphically invertible matrix. In this case the differential form $\omega = dY(z,a)Y^{-1}(z,a)$ is given by

$$\omega = \sum_{i=1}^{n} \frac{A_i'(a)}{z-a_i}\, d(z-a_i) + \sum_{k=1}^{n} \gamma_k(a) da_k. \tag{8}$$

This isomonodromic deformation is not normalized. It is clear that this deformation is reduced to the Schlesinger deformation by

$$Y_{Schl}(z,a) = \Gamma^{-1}(a) Y(z,a). \tag{9}$$

However, there exist isomonodromic deformations given by differential 1-forms different from (5) and (8). A. A. Bolibruch gave examples of such deformations and obtained a general form of the isomonodromic deformation.

Definition 2.3 ([3, 4])**.** Let $\lambda_1^i, \ldots, \lambda_p^i$ be the eigenvalues of the matrix A_i of the Fuchsian system (3). A singular point a_i is called resonant if there exist at least two non-equal eigenvalues of A_i such that their difference is a natural number. A number

$$r_i = \max_{k\neq j, |\lambda_k^i - \lambda_j^i| \in \mathbf{N}} |\lambda_k^i - \lambda_j^i|$$

is called a maximal i-resonance of the system.

Theorem 2.2 ([3, 4])**.** *Any matrix-valued differential 1-form ω on $\bar{\mathbf{C}} \times D(a^0) \setminus \cup_{i=1}^{n} \{z - a_i = 0\}$ which defines isomonodromic deformation of the Fuchsian system* (3) *is given by*

$$\omega = \sum_{i=1}^{n} \frac{A_i(a)}{z-a_i}\, d(z-a_i) + \sum_{k=1}^{n} \gamma_k(a) da_k + \sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{m=1}^{r_l} \frac{\gamma_{m,k,l}(a)}{(z-a_l)^m}\, da_k, \tag{10}$$

where $\gamma_{m,k,l}(a), \gamma_k(a)$ are holomorphic in $D(a^0)$ and r_l is a maxmal l-resonance of system (3) *for $a = a^0$.*

We remark that the last terms may be non-zero only if system (3) has some resonant singularities.

The famous example [6, 3, 4] of Bolibruch of the non-Schlesinger normalized isomonodromic deformation is given as follows. The family of Fuchsian systems

$$\frac{dy}{dz} = \left(\begin{pmatrix} 1 & 0 \\ -\frac{2a}{a^2-1} & 0 \end{pmatrix} \frac{1}{z+a} + \begin{pmatrix} 0 & -6a \\ 0 & -1 \end{pmatrix} \frac{1}{z} + \begin{pmatrix} 2 & 3+3a \\ \frac{1}{1+a} & -1 \end{pmatrix} \frac{1}{z-1} + \begin{pmatrix} -3 & -3+3a \\ \frac{1}{a-1} & 2 \end{pmatrix} \frac{1}{z+1} \right) y$$

is isomonodromic with the differential 1-form ω given by

$$\omega = \begin{pmatrix} 1 & 0 \\ -\frac{2a}{a^2-1} & 0 \end{pmatrix} \frac{d(z+a)}{z+a} + \begin{pmatrix} 0 & -6a \\ 0 & -1 \end{pmatrix} \frac{dz}{z} + \begin{pmatrix} 0 & 0 \\ \frac{2a}{a^2-1} & 0 \end{pmatrix} \frac{da}{z+a} + \begin{pmatrix} 2 & 3+3a \\ \frac{1}{1+a} & -1 \end{pmatrix} \frac{d(z-1)}{z-1} + \begin{pmatrix} -3 & -3+3a \\ \frac{1}{a-1} & 2 \end{pmatrix} \frac{d(z+1)}{z+1}.$$

Here $z = -a$ and $z = 0$ are resonant singular points. The deformation is normalized (there is no term of the form $\gamma(a)da$) and it cannot be reduced to the Schlesinger deformation (there is a term $\gamma(a)da/(z+a)$). In general it is difficult to find explicit examples of the non-Schlesinger deformations (see, for instance, [1, 2] for more examples and a discussion).

3. Middle convolution. Middle convolution is an operation on tuples of residue matrices of a Fuchsian system introduced by S. Reiter and M. Dettweiler [7, 8]. For a given parameter $\mu \in \mathbf{C}$

one defines residue matrices of dimension $pn \times pn$ which are partitioned into blocks and have only one non-zero block consisting of the initial residue matrices and the parameter μ. By finding invariant subspaces and reducing the size of the matrices one gets a new Fuchsian system with the same singularities but with new residue matrices. This operation can be realized as an analytic operation for solutions (Euler transformation). Note that the size of matrices in the final system depends on the choice of the parameter μ. See [7, 9] for more details and explicit expressions.

The properties of middle convolution for Fuchsian systems, its generalization to systems with irregular singularities and q-difference systems are studied by many Japanese authors (e.g., Y. Haraoka, K. Hiroe, H. Kawakami, T. Oshima, K. Takemura, H. Sakai, M. Yamaguchi, D. Yamakawa and many others). Since middle convolution preserves Schlesinger deformation equations [9], it is natural to ask what happens to Bolibruch's non-Schlesinger isomonodromic deformations under middle convolution.

4. Outline of the main results. In [2] we construct explicit examples that middle convolution does not in general preserve non-Schlesinger deformation. Our examples are based on the modifications of the Bolibruch example. Note that it is very important and of interest to specialists in deformation theory to find new explicit examples of non-Schlesinger isomonodromic deformations because of the difficulty to write down the corresponding differential 1-form ω [1]. Our explicit examples show that under middle convoution the resonance condition may appear or disappear. Moreover, the maximal i-resonance of a system may change.

It is easy to show that if we apply middle convolution with $\mu = 0$ to Bolibruch's example and get a new isomonodromic (2×2)-family, which is non-Schlesinger again and non-normalized. It is an expected result because of the properties of middle convolution and isomonodromic deformations. Therefore, we modify the Bolibruch example.

We found an explicit example which shows that applying middle convolution with some μ, $\mu \neq 0$, to a certain non-Schlesinger isomonodromic (2×2)-family with five singular points, the resulting family cannot be included in any Bolibruch's non-Schlesinger isomonodromic deformation because of Theorem 2.2. The resulting family does not also satisfy the Schlesinger equations and it cannot be transformed to a Schlesinger isomonodromic deformation by a transformation $y = \Gamma(a)\tilde{y}$ for any holomorphic matrix-value function $\Gamma(a)$. The initial system has two resonant Fuchsian points with the maximal resonances both equal to 1. The parameter μ of middle convolution was chosen to break old resonances without producing new ones, i.e., the resulting system is non-resonant.

Finally, we were able to demonstrate explicitly the opposite case. We can start again with the non-Schlesinger isomonodromic (2×2)-family with four singularities. This family has two resonant singularities with the maximal resonances 1 and 4 respectively. After applying middle convolution with $\mu \notin \frac{1}{2}\mathbf{Z}$, $\mu \neq 0$ we get a new (5×5)-family with one resonant point with the maximal resonances equal to 1. This family is the non-Schlesinger isomonodromic deformation and it cannot be reduced to any Schlesinger isomonodromic deformation. To prove this we presented a differential 1-form ω for which conditions in Theorem 2.1 are fulfilled. This form contains terms which cannot be elliminated by a holomorphic gauge transformation.

Due to space limitations, the details are published separately [2].

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