A note on the Bloch-Tamagawa space and Selmer groups

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Abstract: For any abelian variety A over a number field, we construct an extension of the Tate-Shafarevich group by the Bloch-Tamagawa space using the recent work of Lichtenbaum and Flach. This gives a new example of a Zagier sequence for the Selmer group of A.

Key words: Abelian varieties; L-functions; Tamagawa numbers; Selmer groups.

Introduction. Let A be an abelian variety over a number field F and $A^\vee$ its dual. Birch and Swinnerton-Dyer, interested in defining the Tamagawa number $\tau(A)$ of A, were led to their celebrated conjecture [2, Conjecture 0.2] for the L-function $L(A,s)$ (of A over F) which predicts both its order r of vanishing and its leading term c_A at $s = 1$. The difficulty in defining $\tau(A)$ directly is that the adelic quotient $\frac{A(\mathbf{A}_F)}{A(F)}$ is Hausdorff only when $r = 0$, i.e., $A(F)$ is finite. Bloch [2] has introduced a semiabelian variety G over F such that $G(F)$ is discrete and cocompact in $G(\mathbf{A}_F)$ [2, Theorem 1.10] and famously proved [2, Theorem 1.17] that the Tamagawa number conjecture (4) for G is equivalent to the Birch-Swinnerton-Dyer conjecture for A over F. Observe that G is not a linear algebraic group.

The aim of this short note is to indicate a functorial construction of a locally compact group Y_A given as

$$0 \to X_A \to Y_A \to Ш(A/F) \to 0, \tag{1}$$

an extension of $Ш(A/F)$ by X_A. The compactness of Y_A is clearly equivalent to the finiteness of $Ш(A/F)$. This would be straightforward if $G(L)$ were discrete in $G(\mathbf{A}_L)$ for all finite extensions L of F. But this is not true (Lemma 4):

$$\frac{G(\mathbf{A}_L)}{G(L)}$$

is not Hausdorff in general.

The very simple idea for the construction of Y_A is: *Yoneda's lemma*. Namely, consider the category of topological $\mathcal{G}$-modules as a subcategory of the classifying topos $B\mathcal{G}$ of $\mathcal{G}$ (continuous cohomology of a topological group $\mathcal{G}$, as in S. Lichtenbaum [10], M. Flach [5]).

D. Zagier [18] has pointed out that the Selmer groups $\mathrm{Sel}_m(A/F)$ (5) can be obtained from certain two-extensions (6) of $Ш(A/F)$ by $A(F)$; these we call Zagier sequences. We show how Y_A provides a new natural Zagier sequence. In particular, this shows that Y_A is not a split sequence.

Bloch's construction has been generalized to one-motives; it led to the Bloch-Kato conjecture on Tamagawa numbers of motives [3]; it is also closely related to Scholl's method of relating non-critical values of L-functions of pure motives to critical values of L-functions of mixed motives [9, p. 252] [13, 14].

Notations. We write $\mathbf{A} = \mathbf{A}_f \times \mathbf{R}$ for the ring of adeles over $\mathbf{Q}$; here $\mathbf{A}_f = \hat{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{Q}$ is the ring of finite adeles. For any number field K, we let $\mathcal{O}_K$ be the ring of integers, $\mathbf{A}_K$ denote the ring of adeles $\mathbf{A} \otimes_{\mathbf{Q}} K$ over K; write $\mathbf{I}_K$ for the ideles. Let $\bar{F}$ be a fixed algebraic closure of F and write $\Gamma = \mathrm{Gal}(\bar{F}/F)$ for the Galois group of F. For any abelian group P and any integer $m > 0$, we write P_m for the m-torsion subgroup of P. A topological abelian group is Hausdorff.

Construction of Y_A. This will use the continuous cohomology of Γ via classifying spaces [10, 5] to which we refer for a detailed exposition.

For each field L with $F \subset L \subset \bar{F}$, the group $G(\mathbf{A}_L)$ is a locally compact group. If L/F is Galois, then

$$G(\mathbf{A}_L)^{\mathrm{Gal}(L/F)} = G(\mathbf{A}_F).$$

So

$$\mathbf{E} = \lim_{\rightarrow} G(\mathbf{A}_L),$$

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the direct limit of locally compact abelian groups, is equipped with a continuous action of $\Gamma = \Gamma_F$. The natural map

$$E := G(\bar{F}) \hookrightarrow \mathbf{E}$$

is Γ_F-equivariant. Though the subgroup $G(F) \subset G(\mathbf{A}_F)$ is discrete, the subgroup

$$E \subset \mathbf{E}$$

fails to be discrete; this failure happens at finite level (see Lemma 4 below). The non-Hausdorff nature of the quotient

$$\mathbf{E}/E$$

directs us to consider the classifying space/topos.

Let *Top* be the site defined by the category of (locally compact) Hausdorff topological spaces with the open covering Grothendieck topology (as in the "gros topos" of [5, §2]). Any locally compact abelian group M defines a sheaf yM of abelian groups on *Top*; this (Yoneda) provides a fully faithful embedding of the (additive, but not abelian) category *Tab* of locally compact abelian groups into the (abelian) category $\mathcal{T}ab$ of sheaves of abelian groups on *Top*. Write $\mathcal{T}op$ for the category of sheaves of sets on *Top* and let $y : Top \to \mathcal{T}op$ be the Yoneda embedding. A generalized topology on a given set S is an object F of $\mathcal{T}op$ with $F(*) = S$.

For any (locally compact) topological group $\mathcal{G}$, its classifying topos $B\mathcal{G}$ is the category of objects F of $\mathcal{T}op$ together with an action $y\mathcal{G} \times F \to F$. An abelian group object F of $B\mathcal{G}$ is a sheaf on $\mathcal{T}op$, together with actions $y\mathcal{G}(U) \times F(U) \to F(U)$, functorial in U; we write $H^i(\mathcal{G}, F)$ (objects of $\mathcal{T}ab$) for the continuous/topological group cohomology of $\mathcal{G}$ with coefficients in F. These arise from the global section functor

$$B\mathcal{G} \to \mathcal{T}ab, \quad F \mapsto F^{y\mathcal{G}}.$$

Details for the following facts can be found in [5, §3] and [10].

(a) (Yoneda) Any topological $\mathcal{G}$-module M provides an (abelian group) object yM of $B\mathcal{G}$; see [10, Proposition 1.1].

(b) If $0 \to M \to N$ is a map of topological $\mathcal{G}$-modules with M homeomorphic to its image in N, then the induced map $yM \to yN$ is injective [5, Lemma 4].

(c) Applying Propositions 5.1 and 9.4 of [5] to the profinite group Γ and any continuous Γ-module M provide an isomorphism

$$H^i(\Gamma, yM) \simeq H^i_{cts}(\Gamma, M)$$

between this topological group cohomology and the continuous cohomology (computed via continuous cochains). This is also proved in [10, Corollary 2.4].

For any map $f : M \to N$ of topological abelian groups, the cokernel of $yf : yM \to yN$ is well-defined in $\mathcal{T}ab$ even if the cokernel of f does not exist in *Tab*. If f is a map of topological $\mathcal{G}$-modules, then the cokernel of the induced map $yf : yM \to yN$, a well-defined abelian group object of $B\mathcal{G}$, need not be of the form yP.

The pair of topological Γ-modules $E \hookrightarrow \mathbf{E}$ gives rise to a pair $yE \hookrightarrow y\mathbf{E}$ of objects of $B\Gamma$. Write $\mathcal{Y}$ for the quotient object $\frac{y\mathbf{E}}{yE}$. As $\mathbf{E}/E$ is not Hausdorff (Lemma 4), $\mathcal{Y}$ is not yN for any topological Γ-module N.

Definition 1. We set $\mathcal{Y}_A = H^0(\Gamma, \mathcal{Y}) \in \mathcal{T}ab$.

Our main result is the

Theorem 2.

(i) *$\mathcal{Y}_A$ is the Yoneda image yY_A of a Hausdorff locally compact topological abelian group Y_A.*

(ii) *X_A is an open subgroup of Y_A.*

(iii) *The group Y_A is compact if and only if Ш(A/F) is finite. If Y_A is compact, then the index of X_A in Y_A is equal to $\#$Ш(A/F).*

As Ш(A/F) is a torsion discrete group, the topology of Y_A is determined by that of X_A.

Proof of Theorem 2. The basic point is the proof of (iii). From the exact sequence

$$0 \to yE \to y\mathbf{E} \to \mathcal{Y} \to 0$$

of abelian objects in $B\Gamma$, we get a long exact sequence (in $\mathcal{T}ab$)

$$0 \to H^0(\Gamma, yE) \to H^0(\Gamma, y\mathbf{E}) \to$$
$$\to H^0(\Gamma, \mathcal{Y}) \to H^1(\Gamma, yE) \xrightarrow{j} H^1(\Gamma, y\mathbf{E}) \to \cdots.$$

We have the following identities of topological groups: $H^0(\Gamma, yE) = yG(F)$ and $H^0(\Gamma, y\mathbf{E}) = yG(\mathbf{A}_F)$. Thus, it suffices to identify $\mathrm{Ker}(j)$ as yШ(A/F). Let $\mathbf{E}^\delta$ denote $\mathbf{E}$ endowed with the discrete topology; the identity map on the underlying set provides a continuous Γ-equivariant map $\mathbf{E}^\delta \to \mathbf{E}$. Since E is a discrete Γ-module, the inclusion $E \to \mathbf{E}$ factorizes via $\mathbf{E}^\delta$. By item (c) above, $\mathrm{Ker}(j)$ is isomorphic to the Yoneda image of the kernel of the composite map

$$H^1_{cts}(\Gamma, E) \xrightarrow{j'} H^1_{cts}(\Gamma, \mathbf{E}^\delta) \xrightarrow{k} H^1_{cts}(\Gamma, \mathbf{E}).$$

Since E and $\mathbf{E}^\delta$ are discrete Γ-modules, the map j' identifies with the map of ordinary Galois cohomology groups

$$H^1(\Gamma, E) \xrightarrow{j''} H^1(\Gamma, \mathbf{E}^\delta).$$

The traditional definition [2, Lemma 1.16] of $Ш(G/F)$ is as $\mathrm{Ker}(j'')$. As

$$Ш(A/F) \simeq Ш(G/F)$$

[2, Lemma 1.16], to prove Theorem 2, all that remains is the injectivity of k. This is straightforward from the standard description of H^1 in terms of crossed homomorphisms: if $f : \Gamma \to \mathbf{E}^\delta$ is a crossed homomorphism with kf principal, then there exists $\alpha \in \mathbf{E}$ with $f : \Gamma \to \mathbf{E}$ satisfies

$$f(\gamma) = \gamma(\alpha) - \alpha \quad \gamma \in \Gamma.$$

This identity clearly holds in both $\mathbf{E}$ and $\mathbf{E}^\delta$. Since the Γ-orbit of any element of $\mathbf{E}$ is finite, the left hand side is a continuous map from Γ to $\mathbf{E}^\delta$. Thus, f is already a principal crossed (continuous) homomorphism. So k is injective, finishing the proof of Theorem 2. □

Remark 3. The proof above shows: If every element of a topological Γ-module N has open stabilizer, then the natural map $H^1(\Gamma, N^\delta) \to H^1(\Gamma, N)$ is injective.

Bloch's construction of G [2, 11]. Write $A^\vee(F) = B \times$ finite. By the Weil-Barsotti formula,

$$\mathrm{Ext}^1_F(A, \mathbf{G}_m) \simeq A^\vee(F).$$

Every point $P \in A^\vee(F)$ determines a semi-abelian variety G_P which is an extension of A by $\mathbf{G}_m$. Let G be the semiabelian variety determined by B:

$$0 \to T \to G \to A \to 0, \tag{2}$$

an extension of A by the torus $T = Hom(B, \mathbf{G}_m)$. The semiabelian variety G is the Cartier dual [4, §10] of the one-motive

$$[B \to A^\vee].$$

The sequence (2) provides (via Hilbert Theorem 90) [2, (1.4)] the following exact sequence

$$0 \to \frac{T(\mathbf{A}_F)}{T(F)} \to \frac{G(\mathbf{A}_F)}{G(F)} \to \frac{A(\mathbf{A}_F)}{A(F)} \to 0. \tag{3}$$

It is worthwhile to contemplate this mysterious sequence: the first term is a Hausdorff, non-compact group and the last is a compact non-Hausdorff group, but the middle term is a compact Hausdorff group!

Lemma 4. *For any field L with $F \subset L \subset \bar{F}$, the group $G(L)$ is a discrete subgroup of $G(\mathbf{A}_L)$ if and only if $A(K) \subset A(L)$ is of finite index.*

Proof. Pick a subgroup $C \simeq \mathbf{Z}^s$ of $A^\vee(L)$ such that $B \times C$ has finite index in $A^\vee(L)$. The Bloch semiabelian variety G' over L determined by $B \times C$ is an extension of A by $T' = Hom(B \times C, \mathbf{G}_m)$. One has an exact sequence $0 \to T'' \to G' \to G \to 0$ defined over L where $T'' = Hom(C, \mathbf{G}_m)$ is a split torus of dimension s. Consider the commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \frac{T''(\mathbf{A}_L)}{T''(L)} & = & \frac{T''(\mathbf{A}_L)}{T''(L)} & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{T'(\mathbf{A}_L)}{T'(L)} & \longrightarrow & \frac{G'(\mathbf{A}_L)}{G'(L)} & \longrightarrow & \frac{A(\mathbf{A}_L)}{A(L)} \\
 & & \downarrow & & \downarrow & & \| \\
0 & \longrightarrow & \frac{T(\mathbf{A}_L)}{T(L)} & \longrightarrow & \frac{G(\mathbf{A}_L)}{G(L)} & \longrightarrow & \frac{A(\mathbf{A}_L)}{A(L)} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0.
\end{array}$$

The proof of surjectivity in the columns follows Hilbert Theorem 90 applied to T'' [2, (1.4)]. The Bloch-Tamagawa space $X'_A = \frac{G'(\mathbf{A}_L)}{G'(L)}$ for A over L is compact and Hausdorff; its quotient by

$$\frac{T''(\mathbf{A}_L)}{T''(L)} = \left(\frac{\mathbf{I}_L}{L^*}\right)^s$$

is $\frac{G(\mathbf{A}_L)}{G(L)}$. The quotient is Hausdorff if and only if $s = 0$. □

A more general form of Lemma 4 is implicit in [2]: For any one-motive $[N \xrightarrow{\phi} A^\vee]$ over F, write V for its Cartier dual (a semiabelian variety), and put

$$X = \frac{V(\mathbf{A}_F)}{V(F)}.$$

Then X is compact if and only if $\mathrm{Ker}(\phi)$ is finite; X is Hausdorff if and only if the image of ϕ has finite index in $A^\vee(F)$.

Tamagawa numbers. Let H be a semisimple algebraic group over F. Since $H(F)$ embeds discretely in $H(\mathbf{A}_F)$, the adelic space $X_H = \frac{H(\mathbf{A}_F)}{H(F)}$ is

Hausdorff. The Tamagawa number $\tau(H)$ is the volume of X_H relative to a canonical (Tamagawa) measure [15]. The Tamagawa number theorem [8, 1] (which was formerly a conjecture) states

$$\tau(H) = \frac{\#\mathrm{Pic}(H)_{\mathrm{torsion}}}{\#\text{Ш}(H)} \tag{4}$$

where $\mathrm{Pic}(H)$ is the Picard group and Ш(H) the Tate-Shafarevich set of H/F (which measures the failure of the Hasse principle). Taking $H = SL_2$ over $\mathbf{Q}$ in (4) recovers Euler's result

$$\zeta(2) = \frac{\pi^2}{6}.$$

The above formulation (4) of the Tamagawa number theorem is due to T. Ono [12, 17] whose study of the behavior of τ under an isogeny explains the presence of $\mathrm{Pic}(H)$, and reduces the semisimple case to the simply connected case. The original form of the theorem (due to A. Weil) is that $\tau(H) = 1$ for split simply connected H. The Tamagawa number theorem (4) is valid, more generally, for any connected linear algebraic group H over F. The case $H = \mathbf{G}_m$ becomes the Tate-Iwasawa [16, 7] version of the analytic class number formula: the residue of the zeta function $\zeta(F, s)$ is the volume of the (compact) unit idele class group $\mathbf{J}_F^1$ of F.

Zagier extensions [18]**.** The m-Selmer group $\mathrm{Sel}_m(A/F)$ (for $m > 0$) fits into an exact sequence

$$0 \to \frac{A(F)}{mA(F)} \to \mathrm{Sel}_m(A/F) \to \text{Ш}(A/F)_m \to 0. \tag{5}$$

D. Zagier [18, §4] has pointed out that while the m-Selmer sequences (5) (for all $m > 1$) cannot be induced by a sequence (an extension of Ш(A/F) by $A(F)$)

$$0 \to A(F) \to\ ?\ \to \text{Ш}(A/F) \to 0,$$

they can be induced by an exact sequence of the form

$$0 \to A(F) \to \mathcal{A} \to \mathcal{S} \to \text{Ш}(A/F) \to 0 \tag{6}$$

and gave examples of such (Zagier) sequences. Combining (1) and (3) above provides the following natural Zagier sequence

$$0 \to A(F) \to A(\mathbf{A}_F) \to \frac{Y_A}{T(\mathbf{A}_F)} \to \text{Ш}(A/F) \to 0.$$

Write $A(\mathbf{A}_{\bar{F}})$ for the direct limit of the groups $A(\mathbf{A}_L)$ over all finite subextensions $F \subset L \subset \bar{F}$. The previous sequence discretized (neglect the topology) becomes

$$0 \to A(F) \to A(\mathbf{A}_F) \to \left(\frac{A(\mathbf{A}_{\bar{F}})}{A(\bar{F})}\right)^{\Gamma} \to \text{Ш}(A/F) \to 0.$$

Remark 5. (i) For an elliptic curve E over F, Flach has indicated how to extract a canonical Zagier sequence via $\tau_{\geq 1}\tau_{\leq 2}R\Gamma(S_{et}, \mathbf{G}_m)$ from any regular arithmetic surface $S \to \mathrm{Spec}\,\mathcal{O}_F$ with $E = S \times_{\mathrm{Spec}\,\mathcal{O}_F} \mathrm{Spec}\,F$.

(ii) It is well known that the class group $\mathrm{Pic}(\mathcal{O}_F)$ is analogous to Ш(A/F) and the unit group $\mathcal{O}_F^\times$ is analogous to $A(F)$. Iwasawa [6, p. 354] proved that the compactness of $\mathbf{J}_F^1$ is equivalent to the two basic finiteness results of algebraic number theory: (i) $\mathrm{Pic}(\mathcal{O}_F)$ is finite; (ii) $\mathcal{O}_F^\times$ is finitely generated. His result provided a beautiful new proof of these two finiteness theorems. Bloch's result [2, Theorem 1.10] on the compactness of X_A uses the Mordell-Weil theorem ($A(F)$ is finitely generated) and the non-degeneracy of the Néron-Tate pairing on $A(F) \times A^\vee(F)$ (modulo torsion).

Question 6. Can one define directly a space attached to A/F whose compactness implies the Mordell-Weil theorem for A and the finiteness of Ш(A/F)?

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