Néron models for admissible normal functions

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Abstract: For any admissible normal function ν over any dimensional base, we construct by the method of log geometry a Néron model such that ν extends to a section of the model over the boundary. The model is Hausdorff and is a relative log manifold.

Key words: Hodge theory; log geometry; log mixed Hodge structure; Néron model; admissible normal function; non-torsion singularity.

Introduction. Let S be a Hausdorff complex analytic manifold with a normal crossing divisor $S \smallsetminus S^*$, and let ν be an admissible normal function on S^* ([9]) with unipotent local monodromies having torsion singularities at the boundary. ("Having torsion singularities" here means that the underlying local system $L_{\mathbf{Q}}$ of ν splits. See 1.5 for some details.) In [7] §6, we construct by the method of log geometry a Hausdorff relative log manifold J_1 over S which "graphs" ν (this means that ν extends to a section of $J_1 \to S$). We called J_1 *the Néron model* in loc. cit. (Note that the above construction is global over the base S.)

In this paper, we generalize the above construction to the case of non-torsion singularities, i.e., we construct a Hausdorff relative log manifold $J_{1,L_{\mathbf{Q}}}$ over S which graphs a given ν not necessarily with torsion singularities. The space $J_{1,L_{\mathbf{Q}}}$ depends only on $L_{\mathbf{Q}}$ and we call it *a Néron model* for $L_{\mathbf{Q}}$.

Actually, we construct a model not only for an admissible normal function (which is regarded as a variation of mixed Hodge structure (VMHS) whose graded quotients are zero unless the weights are -1 and 0) but also for any admissible VMHS. Moreover, the base S can be any fs log generalized analytic space (which means an object of $\mathcal{B}(\log)$; see 1.1).

We review the several related works. If S is a disk, Green–Griffiths–Kerr [2] constructed a Néron model, which is homeomorphic to our Néron model in [5] 8.2, [7] §6, as proved by Hayama [3].

If S is of higher dimension, Brosnan–Pearlstein–Saito [1] constructed a generalization of the Néron model of Green–Griffiths–Kerr. In [10], Schnell constructed a connected Néron model, and compared it with the construction in [1]. The relationships between theirs and the (connected) Néron model in [7] §6 are not known.

In Section 1, we state the main result. In Section 2, we review a general construction in [7] of a relative log manifold $J_{S,\Sigma}$ from some combinatorial data Σ called weak fan (see 2.6 for the definition of weak fan). In Section 3, we prove the main result. The proof goes roughly as follows. The structure of the proof is parallel to that for the special case in [7] §6. Let ν be an admissible normal function over S. Let σ be the admissible nilpotent cone associated to ν. By virtue of the general machinery in Section 2, the problem is to prove that the translations of all faces of σ by the adjoint action of $G_{\mathbf{Z}}$ form a weak fan. Here $G_{\mathbf{Z}}$ is the group of automorphisms of the lattice. Since $G_{\mathbf{Z}}$ is isomorphic to the semi-direct product of the unipotent part $G_{\mathbf{Z},u}$ and the diagonal part $G'_{\mathbf{Z}}$, it is enough to treat the $G_{\mathbf{Z},u}$-translations and the $G'_{\mathbf{Z}}$-translations. The former case can be treated in the same way as in the special case where $L_{\mathbf{Q}}$ splits. The treatment of the $G'_{\mathbf{Z}}$-translations requires some new arguments.

1. Main results.

1.1. We first review the category $\mathcal{B}(\log)$ in which Néron models live. It was introduced in [8] 3.2.4 as an extended category of the category of fs log analytic spaces.

Let $\mathcal{B}$ (resp. $\mathcal{B}(\log)$) be the category of the local ringed spaces over $\mathbf{C}$ (resp. the ones endowed with a log structure) which are locally isomorphic to a strong subspace of an analytic space (resp. an fs log

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analytic space). See [7] 1.1.2 for the definition of a strong subspace.

In the rest of this section, we fix an object $S \in \mathcal{B}(\log)$ as a base. Our models belong to the category $\mathcal{B}(\log)/S$ of objects of $\mathcal{B}(\log)$ over S and are relative log manifolds in the sense of [7] 1.1.6.

1.2. Let S be an object of $\mathcal{B}(\log)$. Assume that for each $w \in \mathbf{Z}$, we are given a polarized log Hodge structure $H_{(w)}$ on S, which are zero with finite exceptions.

Let $L_{\mathbf{Q}}$ (resp. $L_{\mathbf{Z}}$) be a locally constant sheaf on $S^{\log}$ of finite dimensional $\mathbf{Q}$-vector spaces (resp. finitely generated free $\mathbf{Z}$-modules) endowed with an increasing filtration W. Assume that, locally on S, there is a log mixed Hodge structure (LMH) ([7] 1.3, [4] 2.3, [8] 2.6) H on S such that $H(\mathrm{gr}^W_w) = H_{(w)}$ ($w \in \mathbf{Z}$) and such that there is an isomorphism $H_{\mathbf{Q}} \simeq L_{\mathbf{Q}}$ (resp. $H_{\mathbf{Z}} \simeq L_{\mathbf{Z}}$) of local systems on $S^{\log}$ preserving W.

A morphism in $\mathcal{B}(\log)$ is said to be *strict* if the pullback log structure from the target space is naturally isomorphic to the log structure on the source space.

Let $S^\circ \in \mathcal{B}$ be the underlying space of S.

The main theorem in this paper is the following.

Theorem 1.3. *There is a relative log manifold $J_{1,L_{\mathbf{Q}}}$ (resp. $J_{0,L_{\mathbf{Z}}}$) over S which is strict over S and which represents the following functor on $\mathcal{B}/S^\circ$:*

$\mathcal{B}/S^\circ \ni S' \mapsto$ the set of isomorphism classes of LMH H' on S' with $H'(\mathrm{gr}^W_w) = H_{(w)}|_{S'}$ ($w \in \mathbf{Z}$) satisfying the following condition: Locally on S', there is an isomorphism of local systems $H'_{\mathbf{Q}} \simeq L_{\mathbf{Q}}$ (resp. $H'_{\mathbf{Z}} \simeq L_{\mathbf{Z}}$) on $(S')^{\log}$ preserving W. Here $H_{(w)}|_{S'}$ is the pullback of $H_{(w)}$ by the structure morphism $S' \to S^\circ$, and S' is endowed with the pullback log structure from S.

In the case $L_{\mathbf{Q}} = \bigoplus_w H_{(w),\mathbf{Q}}$ (resp. $L_{\mathbf{Z}} = \bigoplus_w H_{(w),\mathbf{Z}}$) with the evident increasing filtration W, this is [7] Theorem 6.1.1. In this case, we denoted $J_{1,L_{\mathbf{Q}}}$ (resp. $J_{0,L_{\mathbf{Z}}}$) by J_1 (resp. J_0) and called it *the Néron model* (resp. *the connected Néron model*).

Note that in the case where there is a negative integer $w \in \mathbf{Z}$ such that $H_{(k)} = 0$ for $k \neq w, 0$ and $H_{(0)} = \mathbf{Z}$, Theorem 1.3 is reduced to [7] 6.1.1 by the fact that locally on S, a section a of $J_{1,L_{\mathbf{Q}}}$ (resp. $J_{0,L_{\mathbf{Z}}}$) gives an isomorphism $J_1 \xrightarrow{\sim} J_{1,L_{\mathbf{Q}}}$ (resp. $J_0 \xrightarrow{\sim} J_{0,L_{\mathbf{Z}}}$), $x \mapsto x + a$, where $x + a$ is defined by the group law of $\mathcal{E}xt^1(\mathbf{Z}, H_{(w)})$.

Corollary 1.4. *Let S be a complex analytic manifold with a normal crossing divisor $S \smallsetminus S^*$, which gives a log structure on S. Let w be a negative integer. Let $H_{(w)}$ be a polarized log Hodge structure of weight w on S. Let ν be an admissible normal function on S, i.e., an extension of $\mathbf{Z}$ by $H_{(w)}$ in the category of log mixed Hodge structures. Then we have*

$$\begin{aligned}&(\textit{a section of } J_{1,L_{\mathbf{Q}}} \textit{ over } S)\\ &\quad = (\textit{a normal function on } S^* \textit{ which is}\\ &\qquad \textit{admissible with respect to } S \textit{ and}\\ &\qquad \textit{which is underlain by } L_{\mathbf{Q}}).\end{aligned}$$

Here $L_{\mathbf{Q}}$ is the underlying local system of ν.

Remark 1.5. Here we explain a little the term "torsion singularities" in the introduction. In the above corollary, we say that ν has torsion singularities if $L_{\mathbf{Q}}$ splits, i.e., the exact sequence

$$0 \to H_{(w),\mathbf{Q}} \to L_{\mathbf{Q}} \to \mathbf{Q} \to 0$$

of local systems splits. Note that this is equivalent to the vanishing of the canonical element of $(R^1\tau_* H_{(w),\mathbf{Q}})_s = (R^1 j_*(H_{(w),\mathbf{Q}}|_{S \smallsetminus S^*}))_s$ at any point s of S determined by the extension of $\mathbf{Q}$ by $H_{(w),\mathbf{Q}}$ on $S^{\log}$ corresponding to ν. Here j is the inclusion $S \smallsetminus S^* \hookrightarrow S$.

2. Moduli spaces of LMH. In this section, we review the general machinery established in [7] §5.

2.1. To prove 1.3, we may work locally on S. So, in the rest of this paper, we may and will assume that there is an LMH H on S such that $H(\mathrm{gr}^W_w) = H_{(w)}$ ($w \in \mathbf{Z}$) and such that $H_{\mathbf{Q}} \simeq L_{\mathbf{Q}}$ (resp. $H_{\mathbf{Z}} \simeq L_{\mathbf{Z}}$) as local systems on $S^{\log}$ with W.

Further, by [7] 5.1.2, we may and will assume that there are the following data (1)–(5).

(1) The (usual) graded-polarized mixed Hodge data $\Lambda = (H_0, W, (\langle\ ,\ \rangle_w)_w, (h^{p,q})_{p,q})$ (see [7] 2.1.1).

For $A = \mathbf{Z}, \mathbf{Q}, \mathbf{R}$, let G_A be the group of the A-automorphisms of $(H_A, W \cap H_A, (\langle\ ,\ \rangle_w)_w)$.

(2) A sharp fs monoid P.

(3) A homomorphism

$$\begin{aligned}&\Gamma' := \mathrm{Hom}(P^{\mathrm{gp}}, \mathbf{Z})\\ &\quad \to G'_{\mathbf{Z}} := G_{\mathbf{Z}}(\mathrm{gr}^W) = \textstyle\prod_w G_{\mathbf{Z}}(\mathrm{gr}^W_w)\end{aligned}$$

whose image consists of unipotent automorphisms.

(4) A strict morphism $\varphi\colon S \to E'_{\sigma'}$ in $\mathcal{B}(\log)$, where $\sigma' := \mathrm{Hom}(P, \mathbf{R}^{\mathrm{add}}_{\geq 0})$. (See [7] 5.1.1 for the definition of the space $E'_{\sigma'}$ on which there is a

canonical polarized log Hodge structure (PLH) of weight w for any $w \in \mathbf{Z}$.)

(5) An isomorphism ι of PLH for each $w \in \mathbf{Z}$ between $H_{(w)}$ and the pullback of the canonical PLH of weight w on $E'_{\sigma'}$ under the morphism φ in (4).

Let $\Gamma := \Gamma' \times_{G'_{\mathbf{Z}}} G_{\mathbf{Z}}$ and $\Gamma_u := \mathrm{Ker}\,(\Gamma \to \Gamma')$. Then, $\Gamma_u = G_{\mathbf{Z},u}$.

2.2. Before starting the preparation of the proof of Theorem 1.3, we outline its proof.

The structure of the proof of 1.3 is the same as the one for the special case [7] 6.1.1 with split monodromy.

Under the setting in 2.1, the general machinery in [7] §5 associates to a weak fan Σ a relative log manifold J_Σ, which graphs H's in 1.3 with local monodromies in Σ. Thus, the problem is reduced to construct suitable weak fans. We will construct weak fans Σ_1 and Σ_0, and show that J_{Σ_1} and J_{Σ_0} satisfy the conditions for $J_{1,L_{\mathbf{Q}}}$ and $J_{0,L_{\mathbf{Z}}}$ of 1.3 respectively.

We explain here how to construct Σ_0. To construct it, we first prove that the set Σ of the Γ_u-translations of all faces of the local monodromy cone of the given local system $L_{\mathbf{Z}}$ makes a weak fan (3.1), whose proof is very parallel to the case of a split local system ([7] 6.2.1). Next, we prove that Σ is, in fact, compatible with Γ, not only with Γ_u. This step (3.4) requires some new arguments.

We proceed to the details.

2.3. First we review the definition of weak fans. As is explained in [7] §2 and in ibid. §5 respectively, there are two formulations of weak fans: an absolute formulation and a relative formulation. In this paper, we work with weak fans in the relative setting, that is, the one in [7] §5.

2.4. For $A = \mathbf{Q}, \mathbf{R}$, let $\mathfrak{g}'_A$ be the Lie algebra associated to $G'_A := G_A(\mathrm{gr}^W)$. Let $\mathfrak{g}_A$ be the Lie algebra associated to G_A.

Let

$$\sigma' \to \mathfrak{g}'_{\mathbf{R}}$$

be the homomorphism of monoids induced by the homomorphism (3) in 2.1.

A *nilpotent cone* is a polyhedral cone σ in the fiber product

$$\sigma' \times_{\mathfrak{g}'_{\mathbf{R}}} \mathfrak{g}_{\mathbf{R}}$$

whose image $\sigma_{\mathfrak{g}}$ in $\mathfrak{g}_{\mathbf{R}}$ is a nilpotent cone in the absolute sense, i.e., the cone generated by finitely many mutually commuting nilpotent elements.

We say that a nilpotent cone σ is *admissible* if $\sigma_{\mathfrak{g}}$ is admissible, i.e., if the action of $\sigma_{\mathfrak{g}}$ on $H_{0,\mathbf{R}}$ is admissible with respect to W ([7] 1.2.4).

We say that σ is *sharp* if it is strictly convex, i.e., $\sigma \cap (-\sigma) = \{0\}$.

2.5. Let σ be a nilpotent cone and let F be an element of the compact dual $\check{D}$ of D. We say that (σ, F) *generates a nilpotent orbit* if $(\sigma_{\mathfrak{g}}, F)$ generates a nilpotent orbit in the absolute sense, i.e., if $\sigma_{\mathfrak{g}}$ is admissible, if any element of $\sigma_{\mathfrak{g}}$ satisfies the Griffiths transversality, and if the all filtrations sufficiently shifted along the direction $\sigma_{\mathfrak{g}}$ are contained in D. See [7] 5.1.5 for the precise definition.

2.6. A *weak fan* Σ in $\sigma' \times_{\mathfrak{g}'_{\mathbf{R}}} \mathfrak{g}_{\mathbf{R}}$ is a non-empty set of sharp rational nilpotent cones satisfying the following conditions (1) and (2).

(1) Any face of an element of Σ also belongs to Σ.

(2) Let $\sigma_1, \sigma_2 \in \Sigma$. Assume that they have a common interior point. Assume also that there is an $F \in \check{D}$ such that (σ_1, F) and (σ_2, F) generate nilpotent orbits. Then $\sigma_1 = \sigma_2$.

A *fan* in $\sigma' \times_{\mathfrak{g}'_{\mathbf{R}}} \mathfrak{g}_{\mathbf{R}}$ is defined, as usual, by replacing (2) with the condition: If $\sigma_1, \sigma_2 \in \Sigma$, then $\sigma_1 \cap \sigma_2$ is a face of σ_1.

Any fan is a weak fan ([7] 5.1.6, cf. [6] 1.7), but the converse is not valid in general.

2.7. Next we review our Néron models and their variants.

Recall $\Gamma = \Gamma' \times_{G'_{\mathbf{Z}}} G_{\mathbf{Z}}$, which acts on $\sigma' \times_{\mathfrak{g}'_{\mathbf{R}}} \mathfrak{g}_{\mathbf{R}}$ via

$$\mathrm{Ad}(\gamma)((x, y)) = (x, \mathrm{Ad}(\gamma_G)y)$$
$$(\gamma \in \Gamma, (x, y) \in \sigma' \times_{\mathfrak{g}'_{\mathbf{R}}} \mathfrak{g}_{\mathbf{R}}).$$

Here γ_G is the image of γ in $G_{\mathbf{Z}}$.

Let the situation be as in 2.1.

Let Σ be a weak fan which is strongly compatible with Γ ([7] 5.1.8).

Let $D_{S,\Sigma}$ be the space of nilpotent orbits in the relative formulation ([7] 5.1.10). The quotient $J_{S,\Sigma} := \Gamma\backslash D_{S,\Sigma}$ is endowed with the structures of an object in $\mathcal{B}(\log)$.

Then, one of the main results 5.2.8 in [7] §5 says that $J_{S,\Sigma}$ is a relative log manifold and that $J_{S,\Sigma}$ is Hausdorff if S is Hausdorff.

2.8. By another main theorem 5.3.3 of [7] §5, the space $J_{S,\Sigma}$ represents the following functor.

Let μ' be the Γ'-level structure on $H(\mathrm{gr}^W) = \bigoplus_w H_{(w)}$ via the isomorphism ι. Then, the functor represented by $J_{S,\Sigma}$ associates to $T \in \mathcal{B}(\log)/S$ the set of isomorphism classes of an LMH H_1 on T with polarized graded quotients satisfying the following conditions (1) and (2).

(1) $\mathrm{gr}^W_w(H_1)$ is isomorphic to the pullback of $H_{(w)}$ for any $w \in \mathbf{Z}$.

(2) For any $t \in T^{\log}$, for any isomorphism $\mu_t\colon H_{1,\mathbf{Z},t} \xrightarrow{\sim} H_0$ ($H_{1,\mathbf{Z},t}$ here denotes the stalk at t of the lattice of H_1) whose gr^W belongs to μ', and for any specialization $\mathcal{O}^{\log}_{T,t} \to \mathbf{C}$, there exists a cone $\sigma \in \Sigma$ such that σ contains the image of the map $\pi_1^+(\tau^{-1}\tau(t)) \to \sigma' \times \mathfrak{g}_{\mathbf{Q}}$ whose first component is induced by φ and whose second component is induced by μ_t, and that $(\sigma, \mu_t(\mathbf{C} \otimes_{\mathcal{O}^{\log}_{T,t}} F_t))$ generates a nilpotent orbit.

Here $\pi_1^+(\tau^{-1}\tau(t)) := \mathrm{Hom}((M_T/\mathcal{O}^\times_T)_{\tau(t)}, \mathbf{N}) \subset \mathrm{Hom}((M_T/\mathcal{O}^\times_T)_{\tau(t)}, \mathbf{Z}) = \pi_1(\tau^{-1}\tau(t))$, and F is the Hodge filtration of H.

3. Proofs. Let the situation be as in 2.1. First, we prove the following.

Theorem 3.1. *Let Υ be a subgroup of $G_{\mathbf{Q},u}$. Let σ be a sharp rational nilpotent cone such that the natural map $\sigma \to \sigma'$ is bijective. Let $\Sigma_\sigma(\Upsilon)$ be the set of cones $\tau_\upsilon := \{(x, \mathrm{Ad}(\upsilon)y) \mid (x,y) \in \tau\} \subset \sigma' \times_{\mathfrak{g}'_{\mathbf{R}}} \mathfrak{g}_{\mathbf{R}}$, where τ ranges over all faces of σ and υ ranges over all elements of Υ. Then $\Sigma_\sigma(\Upsilon)$ is a weak fan.*

Remark 3.2. [7] 6.2.1 is the case where σ is the 0-lift of σ', i.e., the case where the natural projection $\sigma' \times_{\mathfrak{g}'_{\mathbf{R}}} \mathfrak{g}_{\mathbf{R}} \to \mathfrak{g}_{\mathbf{R}}$ sends σ into the image of the canonical injection $\mathfrak{g}'_{\mathbf{R}} \hookrightarrow \mathfrak{g}_{\mathbf{R}}$.

Proof. Though the proof is, more or less, a repetition of that for the case of the split monodromy [7] 6.2.1, we include it for the completeness.

Since $\Sigma_\sigma(\Upsilon)$ is clearly closed under the operation of taking a face, it is enough to examine the following condition (1).

(1) Let τ and ρ be faces of σ. Let π, υ be elements of Υ. Assume that τ_π and ρ_υ have a common interior point N. (Here τ_π and ρ_υ are "τ_υ" defined in the statement of 3.1 with π as υ and ρ as τ, respectively.) Assume that there is an $F \in \check{D}$ such that (τ_π, F) and (ρ_υ, F) generate nilpotent orbits. Then we have $\tau_\pi = \rho_\upsilon$.

First, we prove $\tau = \rho$. Let τ' and ρ' be the images of τ and ρ in σ' respectively. Since the image of N in σ' is a common interior point of τ' and ρ' and since both τ' and ρ' are faces of σ', we have $\tau' = \rho'$. Hence $\tau = \rho$ because $\sigma \to \sigma'$ is bijective by assumption.

Further, to prove (1), we may assume that $\pi = 1$.

Let $N_{\mathfrak{g}}$ be the image of N in $\mathfrak{g}_{\mathbf{R}}$. We claim

$$N_{\mathfrak{g}}\upsilon = \upsilon N_{\mathfrak{g}}. \tag{2}$$

In fact, $\mathrm{gr}^W(N_{\mathfrak{g}}) = \mathrm{gr}^W(\mathrm{Ad}(\upsilon)^{-1}N_{\mathfrak{g}})$ because $\upsilon \in \Upsilon \subset G_{\mathbf{Q},u}$. On the other hand, both $N_{\mathfrak{g}}$ and $\mathrm{Ad}(\upsilon)^{-1}N_{\mathfrak{g}}$ belong to $\tau \subset \sigma$. Hence, by the injectivity of $\sigma \to \sigma'$, they coincide, that is, $N_{\mathfrak{g}} = \mathrm{Ad}(\upsilon)^{-1}N_{\mathfrak{g}}$. The claim (2) follows.

Let $\tau_{\mathfrak{g}}$ be the image of τ in $\mathfrak{g}_{\mathbf{R}}$. Since (τ, F) generates a nilpotent orbit, the action of $\tau_{\mathfrak{g}}$ on $H_{0,\mathbf{R}}$ is admissible with respect to W. Hence, by [7] 2.2.8, the adjoint action of $\tau_{\mathfrak{g}}$ on $\mathfrak{g}_{\mathbf{R}}$ is admissible with respect to the filtration $W\mathfrak{g}_{\mathbf{R}}$ on $\mathfrak{g}_{\mathbf{R}}$ induced by W.

Let $M := M(\tau_{\mathfrak{g}}, W\mathfrak{g}_{\mathbf{R}}) = M(\mathrm{ad}(N_{\mathfrak{g}}), W\mathfrak{g}_{\mathbf{R}})$. Since $\upsilon - 1 \in (W\mathfrak{g}_{\mathbf{R}})_{-1}$ and $\mathrm{ad}(N_{\mathfrak{g}})(\upsilon - 1) = 0$ by (2), we see

$$\upsilon - 1 \in M_{-1} \tag{3}$$

by [7] 1.2.1.3.

Let $h \in \tau_{\mathfrak{g}}$. We prove $h = \upsilon h \upsilon^{-1}$, which implies $\tau = \tau_\upsilon$. By definition of M, we have $\mathrm{ad}(h)M_{-1} \subset M_{-3}$. Hence, by the above (3), $h\upsilon - \upsilon h = h(\upsilon - 1) - (\upsilon - 1)h \in M_{-3}$. Applying υ^{-1} from the right, we have $h - \upsilon h \upsilon^{-1} \in M_{-3}$. (Note that υ^{-1} preserves M by (2).) Since $(\tau_{\mathfrak{g}}, F)$ and $(\tau_{\mathfrak{g},\upsilon} (:= \mathrm{Ad}(\upsilon)(\tau_{\mathfrak{g}})), F)$ generate nilpotent orbits (2.5), the Griffiths transversality implies $h \in F^{-1}\mathfrak{g}_{\mathbf{C}}$ and $\upsilon h \upsilon^{-1} \in F^{-1}\mathfrak{g}_{\mathbf{C}}$, respectively. Hence $h - \upsilon h \upsilon^{-1} \in F^{-1}\mathfrak{g}_{\mathbf{C}}$. Therefore, $h - \upsilon h \upsilon^{-1} \in M_{-3} \cap F^{-1}\mathfrak{g}_{\mathbf{C}} \cap \overline{F}^{-1}\mathfrak{g}_{\mathbf{C}} = \{0\}$. Here we use the fact that $(M, F\mathfrak{g}_{\mathbf{C}})$ is an $\mathbf{R}$-mixed Hodge structure. Thus we have $\tau = \tau_\upsilon$, and (1) is verified. □

3.3. We prove one more lemma before the proof of Theorem 1.3.

Take a subgroup Υ of $G_{\mathbf{Q},u}$. Assume

(1) $G_{\mathbf{Z},u} \subset \Upsilon$.

(2) For any $\upsilon \in \Upsilon$ and $\gamma_1, \gamma_2 \in G_{\mathbf{Z}}$, if $\gamma_1\upsilon\gamma_2$ belongs to $G_{\mathbf{Q},u}$, then it also belongs to Υ.

Both $G_{\mathbf{Z},u}$ and $G_{\mathbf{Q},u}$ are examples of Υ which satisfy (1) and (2).

Assume also the following condition.

(3) For any $\gamma' \in \Gamma'$, there is an element (γ', γ_G) of Γ such that $(\gamma', \log(\gamma_G))$ is in $\sigma_{\mathbf{R}}$.

Lemma 3.4. *Under the above assumptions in 3.3, $\Sigma_\sigma(\Upsilon)$ is strongly compatible with Γ.*

Proof. By [7] Remark in 5.1.8, it is enough to show that $\Sigma := \Sigma_\sigma(\Upsilon)$ is compatible with Γ. Let τ be a face of σ and $\upsilon \in \Upsilon$. Let $\gamma \in \Gamma$. We have to show $\mathrm{Ad}(\gamma)(\tau_\upsilon) \in \Sigma$, where τ_υ is as in 3.1. Since Γ is naturally isomorphic to a semi-direct product of Γ' and $G_{\mathbf{Z},u}$, and since $G_{\mathbf{Z},u}$ is contained in Υ by the property 3.3 (1) of Υ, we may assume that $\gamma \in \Gamma'$. Here we regard Γ' as a subgroup of Γ. We write $\gamma = (\gamma', \gamma'_G)$. Let (γ', γ_G) be the element of Γ which satisfies the condition (3) in 3.3 for this γ'.

Let $v' := \gamma'_G v \gamma_G^{-1}$. Then, the image of v' in $G'_{\mathbf{Q}}$ is trivial so that $v' \in G_{\mathbf{Q},u}$ and $v' \in \Upsilon$ by (2) in 3.3. We have $\gamma(1, v) = (\gamma', \gamma'_G v) = (1, v')(\gamma', \gamma_G)$ in $\Gamma' \times_{G'_{\mathbf{Q}}} G_{\mathbf{Q}}$. Hence, we have $\mathrm{Ad}(\gamma)(\tau_v) = \mathrm{Ad}(v')(\tau_{(\gamma',\gamma_G)})$, where $\tau_{(\gamma',\gamma_G)} := \{(x, \mathrm{Ad}(\gamma_G)y) \mid (x, y) \in \tau\}$. Since $(\gamma', \log(\gamma_G)) \in \sigma_{\mathbf{R}}$, this $\tau_{(\gamma',\gamma_G)}$ is nothing but τ, so that $\mathrm{Ad}(v')(\tau_{(\gamma',\gamma_G)}) = \tau_{v'}$. The compatibility of Σ and Γ follows. □

3.5. Now we prove the main theorem 1.3. Let σ be the local monodromy cone for $L_{\mathbf{Q}}$ (resp. $L_{\mathbf{Z}}$) in 1.3. Let $\Upsilon = G_{\mathbf{Q},u}$ (resp. $G_{\mathbf{Z},u}$). Then, 3.3 (3) is satisfied because the homomorphism $\Gamma' \to G'_{\mathbf{Z}}$ in 2.1 naturally factors through $\Gamma(\sigma)^{\mathrm{gp}} \subset G_{\mathbf{Z}}$.

Let τ be a face of σ. Let $\Gamma'(\tau')$ be the face of $\mathrm{Hom}(P, \mathbf{N})$ corresponding to the image of τ in σ'.

Let

$$\Sigma_0 := \Sigma_\sigma(G_{\mathbf{Z},u}),$$

$$\Sigma_1 := \left\{ \tau_v \in \Sigma_\sigma(G_{\mathbf{Q},u}) \;\middle|\; \begin{array}{l} \Gamma(\tau_v) \to \Gamma'(\tau') \\ \text{is an isomorphism} \end{array} \right\}.$$

Then Σ_1 is a weak fan which is strongly compatible with Γ, and J_{Σ_1} coincides with the open set of $J_{\Sigma_\sigma(G_{\mathbf{Q},u})}$ consisting of all points at which $J_{\Sigma_\sigma(G_{\mathbf{Q},u})} \to S$ is strict.

The theorem 1.3 reduces to the following proposition.

Proposition 3.6. (i) $\Sigma_0 \subset \Sigma_1$. *In particular,* J_{Σ_0} *is an open subspace of* J_{Σ_1}.

(ii) J_{Σ_1} *(resp.* J_{Σ_0}*) has the property of* $J_{1,L_{\mathbf{Q}}}$ *(resp.* $J_{0,L_{\mathbf{Z}}}$*) in* 1.3.

Proof. (i) is easy to see. (ii) follows from [7] 5.2.8 (i). □

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