Heights of motives

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(Communicated by Heisuke HIRONAKA, M.J.A., Feb. 12, 2014)

Abstract: We define the height of a motive over a number field. We show that if we assume the finiteness of motives of bounded height, Tate conjecture for the p-adic Tate module can be proved for motives with good reduction at p.

Key words: Height; motive; Tate conjecture; Hodge theory; p-adic Hodge theory.

0.1. In this paper, we generalize the definition of the height of an abelian variety over number field due to Faltings [2] to a motive over a number field. Here by motive, we mean a pure motive.

We define the height of a motive M over a number field K as the Arakelov degree of the one dimensional $\mathbf{Q}$-vector space

$$L(M)_{\mathbf{Q}} := \otimes_{r\in\mathbf{Z}}\,(\det_{\mathbf{Q}}\,\mathrm{gr}^r M_{dR})^{\otimes r}$$

which is endowed with a metric at the infinite place and an integral structure. Here M_{dR} denotes the de Rham realization of M, and gr^r is that of the Hodge filtration on M_{dR}.

The metric on $L(M)_{\mathbf{Q}}$ is defined by using Hodge theory (see Section 1), and the integral structure of $L(M)_{\mathbf{Q}}$ is defined by using p-adic Hodge theory (see Section 2).

In the case M is the H_1 of an abelian variety A, $\mathrm{gr}^r M_{dR}$ is $Lie(A)$ if $r = -1$ and is 0 if $r \neq 0, -1$, and our definition coincides with the height of Faltings who used the Arakelov degree of $coLie(A)$ (the dual of $Lie(A)$).

Height is a basic notion in number theory. We hope that our generalization of this notion to motives will supply fruitful subjects and interesting problems to number theory.

Details of this paper will be given elsewhere.

1. Hodge theory.

1.1. In this section, for a pure Hodge structure $H = (H_{\mathbf{Z}}, F)$, we define a canonical metric on the one dimensional $\mathbf{C}$-vector space

$$L(H) := \otimes_{r\in\mathbf{Z}}\,(\det_{\mathbf{C}}\,F^r/F^{r+1})^{\otimes r}.$$

1.2. Let $H = (H_{\mathbf{Z}}, F)$ be a pure Hodge structure of weight w. That is, $H_{\mathbf{Z}}$ is a free $\mathbf{Z}$-module of finite rank and F is a descending filtration on $H_{\mathbf{C}} = \mathbf{C}\otimes H_{\mathbf{Z}}$ such that $F^r = H_{\mathbf{C}}$ for $r \ll 0$ and $F^r = 0$ for $r \gg 0$, satisfying

$$H_{\mathbf{C}} = \oplus_{r\in\mathbf{Z}} H_{\mathbf{C}}^{r,w-r} \quad \text{where } H_{\mathbf{C}}^{r,w-r} = F^r \cap \bar{F}^{w-r}.$$

Here $\bar{F}^r$ is the image of F^r under the complex conjugate $H_{\mathbf{C}} \to H_{\mathbf{C}}\,;\, z\otimes h \mapsto \bar{z}\otimes h$ $(z \in \mathbf{C}, h \in H_{\mathbf{Z}})$.

Let

$$L'(H) := \otimes_{r\in\mathbf{Z}}\,(\det_{\mathbf{C}}\,H^{r,w-r})^{\otimes r}.$$

The canonical isomorphism $H^{r,w-r} \xrightarrow{\cong} F^r/F^{r+1}$ induces a canonical isomorphism $L'(H) \xrightarrow{\cong} L(H)$.

1.3. The complex conjugate $H_{\mathbf{C}} \to H_{\mathbf{C}}$ induces an isomorphism

$$\begin{aligned} L'(H) \xrightarrow{\cong} \bar{L}'(H) &:= \otimes_{r\in\mathbf{Z}}\,(\det_{\mathbf{C}}\,H^{w-r,r})^{\otimes r} \\ &= \otimes_{r\in\mathbf{Z}}\,(\det_{\mathbf{C}}\,H^{r,w-r})^{\otimes(w-r)} \end{aligned}$$

where the last $=$ is obtained by replacing r by $w-r$. We have a canonical isomorphism

$$L'(H)\otimes_{\mathbf{C}}\bar{L}'(H) \cong \mathbf{C}\otimes_{\mathbf{Z}}(\det_{\mathbf{Z}}\,H_{\mathbf{Z}})^{\otimes w}$$

as follows:

$$\begin{aligned} &L'(H)\otimes_{\mathbf{C}}\bar{L}'(H) \\ &\quad= (\otimes_{r\in\mathbf{Z}}\,(\det_{\mathbf{C}}\,H^{r,w-r})^{\otimes r}) \\ &\qquad \otimes (\otimes_{r\in\mathbf{Z}}\,(\det_{\mathbf{C}}\,H^{r,w-r})^{\otimes(w-r)}) \\ &\quad= \otimes_{r\in\mathbf{Z}}\,(\det_{\mathbf{C}}\,H^{r,w-r})^{\otimes w} = (\det_{\mathbf{C}}\,H_{\mathbf{C}})^{\otimes w} \\ &\quad= \mathbf{C}\otimes_{\mathbf{Z}}(\det_{\mathbf{Z}}\,H_{\mathbf{Z}})^{\otimes w}. \end{aligned}$$

1.4. Let $s \in L'(H)$. Then we have an element $\bar{s}$ of $\bar{L}'(H)$, and $s\otimes\bar{s}$ is sent to an element ze of $\mathbf{C}\otimes_{\mathbf{Z}}(\det_{\mathbf{Z}}\,H_{\mathbf{Z}})^{\otimes w}$ via the canonical isomorphism, where $z \in \mathbf{C}$ and e is a $\mathbf{Z}$-basis of $(\det_{\mathbf{Z}}\,H_{\mathbf{Z}})^{\otimes w}$. We define $|s| = |z|^{1/2}$.

Via the canonical isomorphism $L'(H) \xrightarrow{\cong} L(H)$, we obtained a metric on $L(H)$.

2000 Mathematics Subject Classification. Primary 14G40; Secondary 14G25, 11G50.

The author is partially supported by an NSF grant.

doi: 10.3792/pjaa.90.49

2. p-adic Hodge theory.

2.1. In this section, let p be a prime number and let K be a finite extension of $\mathbf{Q}_p$. Let T be a free $\mathbf{Z}_p$-module of finite rank endowed with a continuous action of $G_K := \mathrm{Gal}(\bar{K}/K)$ such that $V = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} T$ is de Rham in the sense of Fontaine [3]. The goal of this section is to define a p-adic integral structure (a $\mathbf{Z}_p$-structure) $L_r(T)$ of

$$L_r(V) := \det_{\mathbf{Q}_p}(D_{dR}(V)/D^r_{dR}(V))$$

for each $r \in \mathbf{Z}$.

$L_r(T)$ is defined to be L_0 of the Tate twist $T(r)$. So, we consider $L_0(T)$.

2.2. As in [1], we have an exact sequence

$$0 \to H^0(K,V) \to D_{\mathrm{crys}}(V)$$
$$\to D_{\mathrm{crys}}(V) \oplus D_{dR}(V)/D^0_{dR}(V) \to H^1_f(K,V) \to 0$$

where H^m are Galois cohomology and $H^1_f(K,V) \subset H^1(K,V)$ is as in [1]. The map $D_{\mathrm{crys}}(V) \to D_{\mathrm{crys}}(V)$ is $x \mapsto (1-\varphi)x$ where φ is the Frobenius, and the map $D_{\mathrm{crys}}(V) \to D_{dR}(V)/D^0_{dR}(V)$ is the evident map.

We will define a $\mathbf{Z}_p$-submodule $H^1_{cf}(K,T)$ of $H^1_f(K,T)$ of finite index, and define the p-adic integral structure $L_0(T)$ of $L_0(V)$ as

$$L_0(T) := \det_{\mathbf{Z}_p} H^1_{cf}(K,T) \otimes (\det_{\mathbf{Z}_p} H^0(K,T))^{\otimes -1}$$
$$\subset \det_{\mathbf{Q}_p} H^1_f(K,V) \otimes (\det_{\mathbf{Q}_p} H^0(K,V))^{\otimes -1}$$
$$\cong \det_{\mathbf{Q}_p} D_{dR}(V)/D^0_{dR}(V) \otimes \det_{\mathbf{Q}_p} D_{\mathrm{crys}}(V)$$
$$\otimes (\det_{\mathbf{Q}_p} D_{\mathrm{crys}}(V))^{\otimes -1} \cong \det_{\mathbf{Q}_p} D_{dR}(V)/D^0_{dR}(V)$$

where the first isomorphism is obtained by the above exact sequence and the second isomorphism is obtained by canceling two $D_{\mathrm{crys}}(V)$ via the identity map.

2.3. We describe our motivation of the definition of $H^1_{cf}(K,T)$.

Let A be an abelian variety over K, and let $TA = \prod_\ell T_\ell A$ where ℓ ranges over all prime numbers (including p) and $T_\ell A$ is the ℓ-adic Tate module. Then the Kummer sequences

$$0 \to TA/nTA \to A(\bar{K}) \xrightarrow{n} A(\bar{K}) \to 0$$

for $n \geq 1$ induce connecting homomorphisms $A(K) \to H^1(K, TA/nTA)$, and the inverse limit gives an isomorphism $A(K) \xrightarrow{\cong} H^1_f(K,TA)$ (see [1]).

Let $\mathcal{A}$ be the Néron model of A and let $\mathcal{A}^\circ \subset \mathcal{A}$ be the connected Néron model of A ($\mathcal{A}^\circ$ is the open set of $\mathcal{A}$ obtained from $\mathcal{A}$ by removing all connected components of the special fiber of $\mathcal{A}$ which do not contain the origin). Thus

$$\mathcal{A}^\circ(O_K) \subset \mathcal{A}(O_K) = A(K) \cong H^1_f(K,TA).$$

For our seek of the nice integral structure on the de Rham object, $\mathcal{A}^\circ(O_K)$ is important. We can identify $\mathcal{A}^\circ(O_K)$ as the subgroup $H^1_{cf}(K,TA)$ of $H^1_f(K,TA)$.

Proposition 2.4. *Let ℓ be a prime number, and let T be a free $\mathbf{Z}_\ell$-module of finite rank endowed with a continuous action of* $\mathrm{Gal}(\bar{K}/K)$. *In the case $\ell = p$, assume that $V = \mathbf{Q}_\ell \otimes_{\mathbf{Z}_\ell} T$ is de Rham. Let $a \in H^1_f(K,T)$. For $n \geq 1$, let K_n be the unique unramified extension of K of degree n.*

(1) *The following three conditions are equivalent.*

(a) *For any $n \geq 1$, a belongs to the image of the trace map $H^1_f(K_n,T) \to H^1_f(K,T)$.*

(b) *For any $n \geq 1$, the image of a in $H^1_f(K_{ur},T)$ belongs to the image of $1-\varphi^n : H^1_f(K_{ur},T) \to H^1_f(K_{ur},T)$. Here K_{ur} is the maximal unramified extension of K and φ is the Frobenius of K_{ur}/K.*

(c) *For any n, the map $H^1_f(K_n,T_a) \to H^1_f(K_n,\mathbf{Z}_\ell) = \mathbf{Z}_\ell$ is surjective. Here T_a is the extension of $\mathbf{Z}_\ell$ by T corresponding to a.*

(2) *If $\ell \neq p$, the equivalent conditions* (a)–(c) *are also equivalent to*

(d) *a belongs to the kernel of $H^1(K,T) \to H^1(K_{ur},T)$.*

(3) *If $T = T_\ell A$ for an abelian variety A over K, the equivalent conditions* (a)–(c) *are also equivalent to*

(e) *a belongs to the image of $\mathcal{A}^\circ(O_K)$.*

Concerning (2), note that in the case $\ell \neq p$, $H^1_f(K,T)$ is defined to be the kernel of $H^1(K,T) \to H^1(K_{ur},V)$ which can be a little bigger than the kernel of $H^1(K,T) \to H^1(K_{ur},T)$.

2.5. We define $H^1_{cf}(K,T)$ to be the subgroup of $H^1_f(K,T)$ consisting of all elements satisfying the equivalent conditions in 2.4 (1) (we take $\ell = p$). We call it the connected finite part of $H^1(K,T)$. We can prove that $H^1_f(K,T)/\, H^1_{cf}(K,T)$ is finite.

2.6. Let A be an abelian variety over K and let $T = T_pA$. Then we have

$$L_0(T) = \det_{\mathbf{Z}_p} \mathrm{Lie}(\mathcal{A}) = \det_{\mathbf{Z}_p} \mathrm{Lie}(\mathcal{A}^\circ).$$

2.7. Assume K is unramifed over $\mathbf{Q}_p$, V is crystalline, and there is $a \in \mathbf{Z}$ such that $D^a_{dR}(V) = D_{dR}(V)$ and $D^{a+p-1}_{dR} = 0$. Then there is an O_K-lattice D of $D_{dR}(V) = D_{\mathrm{crys}}(V)$ corresponding to T which satisfies $D = \sum_{i \in \mathbf{Z}} p^{-i}\varphi D^i$ where $D^i = D \cap D^i_{dR}(V)$.

(T is constructed from D by the method of Fontaine-Laffaille.) In this case, we have

$$H^1_f(K,T) = H^1_{cf}(K,T)$$

and we have an exact sequence

$$0 \to H^0(K,T) \to D^0 \stackrel{1-\varphi}{\to} D \to H^1_{cf}(K,T) \to 0.$$

In the case $T = H^m_{et}(X_{\bar{K}}, \mathbf{Z}_p)(r)/(\text{torsion})$ for a proper smooth scheme X over O_K with $m \in \mathbf{Z}$ such that $m \leq p-2$ and with $r \in \mathbf{Z}$, if Y denotes the special fiber of X, then $D = H^m_{dR}(X/O_K)/(\text{torsion}) = H^m_{\text{crys}}(Y)/(\text{torsion})$, $(D^i)_i$ coincides with r-twist of the Hodge filtration of $H^m_{dR}(X/O_K)/(\text{torsion})$, and φ of D coincides with $p^{-r}\varphi$ of $H^m_{\text{crys}}(Y)/(\text{torsion})$.

2.8. Let K' be a finite extension of $\mathbf{Q}_p$ contained in K and let T' be the Weil restriction of T to K' (that is, the induced representation of $G_{K'}$ obtained from T). Then $H^1_f(K,T) = H^1_f(K',T')$, $H^1_{cf}(K,T) = H^1_{cf}(K',T')$, and $L_0(T) = L_0(T')$.

This corresponds to the fact that the Weil restriction of $\mathcal{A}$ (resp. $\mathcal{A}^\circ$) to $O_{K'}$ is the Néron model (resp. connected Néron model) of the Weil restriction A' of A to K'.

3. Heights of motives.

3.1. Let $\mathbf{A}_f$ be the ring of finite adeles of $\mathbf{Q}$.

Let k be a field of characteristic 0. For a motive M over k (this means the usual pure motive with $\mathbf{Q}$-coefficients), let $M_{\mathbf{A}_f}$ is the étale realization of M with $\mathbf{A}_f$-coefficients which is endowed with the continuous action of $G_k = \text{Gal}(\bar{k}/k)$.

By a $\mathbf{Z}$-motive over k, we mean a motive M over k endowed with a G_k-stable $\hat{\mathbf{Z}}$-submodule T of $M_{\mathbf{A}_f}$ which is free of finite type over $\hat{\mathbf{Z}}$ such that $\mathbf{A}_f \otimes_{\hat{\mathbf{Z}}} T = M_{\mathbf{A}_f}$.

In this section, we define the height of a $\mathbf{Z}$-motive over a number field.

3.2. To avoid a technical problem, we fix integers a, b such that $a \leq b$, and we define the height of a $\mathbf{Z}$-motive M over a number field K satisfying $M^a_{dR} = M_{dR}$ and $M^b_{dR} = 0$, depending on the choices of a, b.

We define the height of such M as the height of the Weil restriction of M to $\mathbf{Q}$. Hence we consider $\mathbf{Z}$-motives over $\mathbf{Q}$.

3.3. Let $M = (M,T)$ be a $\mathbf{Z}$-motive over $\mathbf{Q}$ such that $M^a_{dR} = M_{dR}$ and $M^b_{dR} = 0$. Let

$$L(M)_{\mathbf{Q}} = \otimes_{r \in \mathbf{Z}} (\det_{\mathbf{Q}} \text{gr}^r M_{dR})^{\otimes r}.$$

We define a metric on $L(M)_{\mathbf{R}} = \mathbf{R} \otimes_{\mathbf{Q}} L(M)_{\mathbf{Q}}$. Let M_B be the Betti realization which is a $\mathbf{Q}$-vector space. Then we have a $\mathbf{Z}$-structure $M_{B,\mathbf{Z}}$ of M_B by

$$M_{B,\mathbf{Z}} = M_B \cap T \subset M_{\mathbf{A}_f}.$$

Via the canonical isomorphism $M_{B,\mathbf{C}} \cong M_{dR,\mathbf{C}}$, $M_{B,\mathbf{C}} = \mathbf{C} \otimes_{\mathbf{Z}} M_{B,\mathbf{Z}}$ has a Hodge filtration $F^r := \mathbf{C} \otimes_{\mathbf{Q}} M^r_{dR}$ $(r \in \mathbf{Z})$, and $H = (M_{B,\mathbf{Z}}, F)$ is a pure Hodge structure. Hence by Section 1, we have a metric on $L(H) = L(M)_{\mathbf{C}}$. By restricting to $L(M)_{\mathbf{R}}$, we have a metric on $L(M)_{\mathbf{R}}$.

Next we define an integral structure $L(M)_{\mathbf{Z}}$ on $L(M)_{\mathbf{Q}}$. For each $r \in \mathbf{Z}$, let

$$L_r(M)_{\mathbf{Q}} = \det_{\mathbf{Q}}(M_{dR}/M^r_{dR}).$$

Then by Section 2, the p-adic component T_p of T regarded as a representation of $G_{\mathbf{Q}_p}$ defines a $\mathbf{Z}_p$-structure $L_r(T_p)$ of $L_r(M)_{\mathbf{Q}_p}$. When p ranges, this gives a $\hat{\mathbf{Z}}$-structure $L_r(M)_{\hat{\mathbf{Z}}}$ of $\mathbf{A}_f \otimes_{\mathbf{Q}} L(M)_{\mathbf{Q}}$. This follows from the fact that if $(M,T) = (H^m(X), H^m_{et}(X_{\bar{\mathbf{Q}}}, \hat{\mathbf{Z}}))$ for a projective smooth scheme X over $\mathbf{Q}$ and if $\mathfrak{X}$ is a projective scheme over $\mathbf{Z}$ such that $X = \mathfrak{X} \otimes \mathbf{Q}$, then $L_r(T_p) = \mathbf{Z}_p \otimes_{\mathbf{Z}} \det_{\mathbf{Z}}(H^m_{dR}(\mathfrak{X}/\mathbf{Z})/F^r H^m_{dR}(\mathfrak{X}/\mathbf{Z}))$ for almost all p by 2.7. Let

$$L_r(M)_{\mathbf{Z}} = L_r(M)_{\mathbf{Q}} \cap L_r(M)_{\hat{\mathbf{Z}}} \subset \mathbf{A}_f \otimes_{\mathbf{Q}} L_r(M)_{\mathbf{Q}}.$$

We define

$$\begin{aligned} L(M)_{\mathbf{Z}} &:= (\otimes_{a<i<b} L_i(M)^{\otimes -1}_{\mathbf{Z}}) \otimes L_b(M)^{\otimes(b-1)}_{\mathbf{Z}} \\ &\subset (\otimes_{a<i<b} L_i(M)^{\otimes -1}_{\mathbf{Q}}) \otimes L_b(M)^{\otimes(b-1)}_{\mathbf{Q}} = L(M)_{\mathbf{Q}}. \end{aligned}$$

The reason why we do not take the simpler definition

$$L(M)_{\mathbf{Z}} := \otimes_{r \in \mathbf{Z}} (L_r(M)^{\otimes -1}_{\mathbf{Z}} \otimes L_{r+1}(M)_{\mathbf{Z}})^{\otimes r}$$

(independently of a and b) is that we are not sure whether this is a finite tensor product.

3.4. For a $\mathbf{Z}$-motive M over $\mathbf{Q}$, we define its multiplicative height $H(M)$ and the logarithmic height $h(M)$ as

$$H(M) = |e|^{-1}, \quad h(M) = -\log(|e|)$$

where e is a $\mathbf{Z}$-basis of $L(M)_{\mathbf{Z}}$ and $| \, |$ is the metric of $L(M)_{\mathbf{R}}$.

3.5. If A is an abelian variety over a number field K, for $M = (H_1(A), T(A))$, the height of A defined by Faltings coincides with our height for the choice $a = -1$ and $b = 1$.

4. Some topics.

4.1. We fix the type $\Phi = (w, (h^r)_{r \in \mathbf{Z}})$ of motives, where w is the weight and $h^r = \dim \mathrm{gr}^r_{dR}$. Take a, b such that $h^r = 0$ unless $a \leq r < b$ and define the height of a **Z**-motive by using these fixed a, b.

The following is a basic conjecture.

Conjecture 4.2. Let K be a number field and let $c > 0$. Then there are only finitely many isomorphism classes of motives over K of type Φ of semi-stable reduction such that $h(M) \leq c$.

4.3. In [2], by using his heights of abelian varieties, Faltings proved the Tate conjecture $\mathbf{Z}_p \otimes Hom(A, B) \xrightarrow{\cong} Hom_{G_K}(T_pA, T_pB)$ for abelian varieties A and B over a number field K. A key point was that the above finiteness is true for abelian varieties. He proved this finiteness by using the fact that his height of an abelian variety essentially coincides with the height of the corresponding point of the moduli space of abelian varieties. For the above general conjecture, the difficulty is that usually there is no moduli space of **Z**-motives of type Φ.

Proposition 4.4. *Let $M = (M, T)$ and $M' = (M', T')$ be **Z**-motives over a number field K of type Φ. Let p be a prime number, and assume that M_p and M'_p are crystalline as representations of G_{K_v} for any place v of K lying over p, p is unramified in $K/\mathbf{Q}$, and that $b \leq a + p - 1$. Assume that Conjecture 4.2 is true. Then*

$$\mathbf{Z}_p \otimes_{\mathbf{Z}} Hom(M, M') \xrightarrow{\cong} Hom_{G_K}(T_p, T'_p).$$

This proposition is proved in the following way.

Lemma 4.5. *Let V be a finite dimensional $\mathbf{Q}_p$-vector space endowed with a continuous action of $G_{\mathbf{Q}}$. Assume that V is de Rham as a representation of $G_{\mathbf{Q}_p}$ and assume that there is an integer $w \in \mathbf{Z}$ such that for almost all prime numbers ℓ, the action of $G_{\mathbf{Q}_\ell}$ on V is unramified and all eigen values of the action of the geometric frobenius of ℓ on V are algebraic numbers whose all conjugates over $\mathbf{Q}$ are of complex absolute value $\ell^{w/2}$. Let*

$$s(V) = \sum_{r \in \mathbf{Z}} r \cdot \dim \mathrm{gr}^r D_{dR}(V),$$
$$t(V) = w \cdot \dim(V)/2.$$

Then we have $s(V) = t(V)$.

Proof. This is reduced to the case $\dim(V) = 1$ by the facts $s(V) = s(\det(V))$ and $t(V) = t(\det(V))$. If $\dim(V) = 1$, V is isomorphic to $\mathbf{Q}_p(m)$ for some integer m as a representation of G_K for some finite extension K of $\mathbf{Q}$, and hence $s(V) = -m = t(V)$. □

If we assume Conjecture 4.2, then by the argument of Faltings in [2], the proof of Proposition 4.4 is reduced to

Proposition 4.6. *Let $M = (M, T)$ be as in the hypothesis of Proposition 4.4. (We do not assume Conjecture 4.2 here.) Let U be a free $\mathbf{Z}_p$-module of finite rank endowed with an action of G_K, and assume that we have a surjective homomorphism $T_p \to U$ which is compatible with the actions of G_K. For $n \geq 0$, let $T^{(n)} := Ker(T \to U/p^nU)$, and let $M^{(n)}$ be the **Z**-motive over K which is the same as M as a **Q**-motive over K but with the Galois representation $T^{(n)}$ over $\hat{\mathbf{Z}}$. Then $h(M^{(n)}) = h(M)$ for any $n \geq 0$.*

Proof. By Weil restriction, we may assume $K = \mathbf{Q}$. Let D be as in 2.7. We have exact sequences $0 \to D^i(T_p^{(n)}) \to D^i(T_p) \to D^i(U)/p^nD^i(U) \to 0$ for all i. From this, we have

$$L(M^{(n)})_{\mathbf{Z}} = p^{ns(V)} L(M)_{\mathbf{Z}}$$

where $V = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} U$. On the other hand,

$$(\det_{\mathbf{Z}} H^{(n)}_{\mathbf{Z}})^{\otimes w} = p^{2nt(V)} (\det_{\mathbf{Z}} H_{\mathbf{Z}})^{\otimes w},$$

where H (resp. $H^{(n)}$) is the Hodge structure associated to M (resp. $M^{(n)}$). Hence

$$H(M)/H(M^{(n)}) = p^{n(s(V) - t(V))} = 1$$

by Lemma 4.5. □

4.7. Many questions arise concerning heights of motives. For example, we have the following analogue of abc conjecture (or Vojta conjecture [4]) for motives. For a motive M over a number field K, let $n(M) = \sum_v \log N(v)$ where v ranges over all finite places of K at which M has bad reduction and $N(v)$ denotes the order of the residue field of v.

Conjecture 4.8. There are constants $c, c' > 0$ such that

$$n(M) \geq c \cdot h(M) - \log(|D_K|) - c' \cdot [K : \mathbf{Q}]$$

for any number field K and any **Z**-motive M over K of type Φ of semi-stable reduction. Here D_K denotes the discriminant of K.

References

[1] S. Bloch and K. Kato, *L*-functions and Tamagawa numbers of motives, in *The Grothendieck Festschrift, Vol. I*, Progr. Math., 86, Birkhäuser Boston, Boston, MA, 1990, pp. 333–400.

[2] G. Faltings, Endlichkeitssätze für abelsche Varietäten über Zahlkörpern, Invent. Math. **73** (1983), no. 3, 349–366.

[3] J.-M. Fontaine, Sur certains types de représentations p-adiques du groupe de Galois d'un corps local; construction d'un anneau de Barsotti-Tate, Ann. of Math. (2) **115** (1982), no. 3, 529–577.

[4] P. Vojta, Diophantine approximation and Nevanlinna theory, in *Arithmetic geometry*, Lecture Notes in Math., 2009, Springer, Berlin, 2011, pp. 111–224.