Determination of a nonlinearity from blow-up time

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Abstract: We study an inverse problem to determine a nonlinearity of an autonomous equation from a blow-up time of solutions of the equation. A local well-posedness of the inverse problem near a nonlinearity of the type $u^{1+\sigma}$, $\sigma > 0$, is established. The paper also suggests that the inverse problem has a good, mathematical structure from a viewpoint of the Wiener-Hopf theory in integral equations.

Key words: Inverse problem; blow-up time; multiplicative Wiener-Hopf.

1. Problem and result. Let $a \in \boldsymbol{R}$ and consider an initial value problem

$$(1.1)\qquad \begin{cases} \dfrac{d^2u}{dt^2} = f(u), & 0 < t < \infty; \\ u(0) = h, & a < h < \infty; \\ \dfrac{du}{dt}(0) = 0, \end{cases}$$

where f is a continuous, positive function on the interval (a, ∞). We impose on f the super-linearity condition

$$(1.2)\qquad \int_b^\infty \frac{du}{\sqrt{\int_b^u f(\xi)d\xi}} < \infty$$

for each $b > a$. A typical example of functions satisfying (1.2) is given by $f(u) = u^{1+\sigma}$, $\sigma > 0$ or those behaving like $u^{1+\sigma}$ as $u \to \infty$.

Because of $f > 0$, the solution of (1.1) is given by an inverse function of $t(u)$ determined by

$$\frac{dt}{du} = \frac{1}{\sqrt{2}\sqrt{\int_h^u f(\xi)d\xi}}, \quad t(h) = 0$$

for $h > a$. Therefore, under the condition (1.2), the solution of (1.1) for each $h \in (a, \infty)$ blows up at the time

$$(1.3)\qquad T_f(h) := \frac{1}{\sqrt{2}} \int_h^\infty \frac{du}{\sqrt{\int_h^u f(\xi)d\xi}}$$

for each $h \in (a, \infty)$. We call T_f the blow-up time function associated with f, and let $\mathcal{B}$ be a map assigning the blow-up time function T_f to f, namely, $\mathcal{B} : f \mapsto T_f$.

We now pose an inverse problem discussed in the present paper:

Problem 1.1. Given a function $T = T(h)$, $a < h < \infty$, determine a nonlinearity f of equation (1.1) so that $\mathcal{B}f = T$.

We assume that $a = 1$ without loss of generality because the shift $\tilde{u} := u - a + 1$, $\tilde{h} := h - a + 1$ and setting $\tilde{f}(\tilde{u}) = f(\tilde{u} + a - 1)$ change (1.1) to

$$\begin{cases} \dfrac{d^2\tilde{u}}{dt^2} = \tilde{f}(\tilde{u}), & 0 < t < \infty; \\ \tilde{u}(0) = \tilde{h}, & 1 < \tilde{h} < \infty; \\ \dfrac{d\tilde{u}}{dt}(0) = 0, \end{cases}$$

where $\tilde{f}$ is a continuous, positive function on the interval $(1, \infty)$. Therefore, throughout the paper, we fix a as $a = 1$. Then Problem 1.1 is equivalent to finding a solution f of

$$(1.4)\qquad \frac{1}{\sqrt{2}} \int_h^\infty \frac{du}{\sqrt{\int_h^u f(\xi)d\xi}} = T(h), \quad 1 < h < \infty,$$

where $T(h)$ is a prescribed, positive function on the interval $(1, \infty)$.

For the typical case $f_0(u) = cu^{1+\sigma}$ with $c, \sigma > 0$, the blow-up time function is calculated as

$$T_0(h) = c' h^{-\frac{\sigma}{2}},$$

where

$$c' = \frac{1}{\sqrt{2c(2+\sigma)}} B\left(\frac{\sigma}{2(2+\sigma)}, \frac{1}{2}\right).$$

In the present paper we discuss Problem 1.1 near this correspondence

$$(1.5)\qquad \mathcal{B} : f_0(u) = cu^{1+\sigma} \mapsto T_0(h) = c' h^{-\frac{\sigma}{2}}.$$

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To define function spaces for f and for T in a unified manner, we introduce a function space. Let $I \subset (0,\infty)$ be an interval, $\alpha \in (0,1]$, $\eta \in \boldsymbol{R}$, and let

$$\mathcal{C}^{\alpha}(I)_{\eta} = \{\phi \in C(I) : |\phi|_{\eta} + |\phi|_{\alpha,\eta} < \infty\}, \tag{1.6}$$

where $|\cdot|_{\eta}$ and $|\cdot|_{\alpha,\eta}$ are semi-norms defined by

$$|\phi|_{\eta} := \sup_{x\in I} \frac{|\phi(x)|}{|x|^{\eta}},$$

$$|\phi|_{\alpha,\eta} := \sup_{\substack{x,y\in I\\ x\neq y}} \frac{|x^{\alpha-\eta}\phi(x) - y^{\alpha-\eta}\phi(y)|}{|x-y|^{\alpha}}.$$

Equipped with the norm $\|\phi\|_{\alpha,\eta} := |\phi|_{\eta} + |\phi|_{\alpha,\eta}$, the space $\mathcal{C}^{\alpha}(I)_{\eta}$ is a Banach space. When I is an open interval such as $I = (1,\infty)$, we omit the bracket of $\mathcal{C}^{\alpha}(I)_{\eta}$ such as $\mathcal{C}^{\alpha}(1,\infty)_{\eta}$.

We can now state our main result (Fig. 1):

Theorem 1.2. *Let α be any number fixed such that $0 < \alpha < \frac{1}{2}$. Then $\mathcal{B}$ maps a sufficiently small neighborhood of f_0 in $\mathcal{C}^{\alpha}(1,\infty)_{1+\sigma}$ homeomorphically onto a neighborhood of T_0 in $\mathcal{C}^{\alpha+\frac{1}{2}}(1,\infty)_{-\frac{\sigma}{2}}$.*

Problem 1.1 is motivated by a use of blowing up solutions to various types of differential equations. We explain it in an aspect of a comparison method. Let $R > 0$ and consider positive C^2-solutions $u(x)$ of the elliptic inequality

$$\Delta u \geq g(u) \quad \text{in} \quad \overline{B(0,R)},$$

where Δ is the N-dimensional Laplace operator, $N \geq 2, B(0,R) = \{x \in \boldsymbol{R}^N : |x| < R\}$, and $g : (0,\infty) \to (0,\infty)$ is a continuous function satisfying $g(+0) \in [0,\infty)$. We want to get upper bounds to $u(x)$ under the assumption that there is a strictly increasing, locally Lipschitz continuous function $g_* : [0,\infty) \to [0,\infty)$ satisfying $0 < g_*(u) \leq g(u)$ in $(0,\infty)$,

$$\int_1^{\infty} G(u)^{-\frac{1}{2}} du < \infty, \text{ and } \int_0^1 G(u)^{-\frac{1}{2}} du = \infty.$$

Here $G(u) := \int_0^u g_*(v)dv$. It is known (see Keller [3], Usami [5]) that there is a positive, monotonically increasing C^2-solution $v(r)$ to the problem

$$\begin{cases} r^{1-N}\dfrac{d}{dr}\left(r^{N-1}\dfrac{dv}{dr}\right) = g_*(v), & 0 < r < R, \\ \dfrac{dv}{dr}(0) = 0, \quad \text{and } v(r) \to \infty & \text{as} \quad r \to R. \end{cases} \tag{1.7}$$

(The constant R is the "blow-up time" of $v(r)$.) So the function $v(|x|)$ satisfies

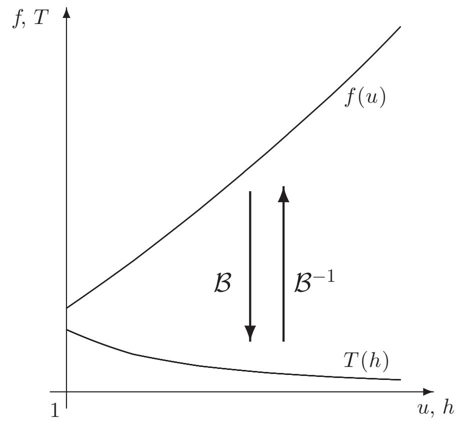

Fig. 1. Local homeomorphism.

$$\begin{cases} \Delta v = g_*(v), & \text{in} \quad B(0,R), \\ \lim\limits_{|x|\to R} v = \infty. \end{cases}$$

Noting the inequality $\Delta(u - v) \geq g_*(u) - g_*(v)$, we can show that $u(x) \leq v(|x|)$ in $B(0,R)$ as in Usami [6]. On the other hand, the monotonicity of $v(r)$ implies that

$$\frac{dv}{dr}(r) \leq g_*(v(r)) \int_0^r \left(\frac{s}{r}\right)^{N-1} ds,$$

and so, $\frac{1}{r}\frac{dv}{dr} \leq \frac{g_*(v)}{N}$. Returning to (1.7), we find that

$$\frac{d^2v}{dr^2} \geq \frac{g_*(v)}{N}.$$

Note that this is an inequality version of the form (1.1). By the same computations as for (1.3) we obtain

$$\int_{v(0)}^{\infty} \frac{dz}{\sqrt{G(z) - G(v(0))}} \geq \sqrt{\frac{2}{N}} R.$$

This is equivalent to $v(0) \leq \tilde{G}^{-1}(\sqrt{\frac{2}{N}}R)$, where $\tilde{G}(u) = \int_u^{\infty} \frac{dz}{\sqrt{G(z)-G(u)}}$. Since $u(x) \leq v(|x|)$ as seen above, we have $u(0) \leq \tilde{G}^{-1}(\sqrt{\frac{2}{N}}R)$. If $x_0 \in B(0,R)$, then $\Delta u \geq g(u)$ in $B(x_0, R - |x_0|)$. Therefore arguing as above, we have an upper estimate

$$u(x_0) \leq \tilde{G}^{-1}\left(\sqrt{\frac{2}{N}}(R - |x_0|)\right).$$

Our success in getting this estimate depended on the existence of the solution $v(r)$ that blows up at the time R. In view of this observation, a general question arises: in what situation a prescribed time becomes the blow-up time. This question leads to Problem 1.1.

The present paper is organized as follows: In Section 2, we give our strategy for proving Theorem 1.2, that is, we show it is enough to prove a mapping $\mathcal{F}$ defined by (2.2) maps a small neighborhood of a constant function c in $\mathcal{C}^{\alpha}(0,1)_0$ homeomorphically onto a neighborhood of $\sqrt{2}\,c'$ in $\mathcal{C}^{\alpha+\frac{1}{2}}(0,1)_0$. To prove this, we apply an inverse mapping theorem to the mapping $\mathcal{F}$. Proposition 3.1 in Section 3 shows that our function spaces setting is appropriate. Proposition 4.1 shows that the Fréchet derivative of $\mathcal{F}$ at c is a homeomorphism of $\mathcal{C}^{\alpha}(0,1)_0$ onto $\mathcal{C}^{\alpha+\frac{1}{2}}(0,1)_0$. The proof of Theorem 1.2 is given at the end of Section 4.

Throughout the paper, we use the notation $A \lesssim B$, which implies that there exists a positive constant M independent of variables of A, B such that $A \le MB$.

2. Reduction. Via a change of variables $x = h^{-1}$, $u = y^{-1}$, equation (1.4) can be recast as

$$\frac{1}{\sqrt{2}}\int_0^x \left(\int_y^x f\Big(\frac{1}{\eta}\Big)\frac{d\eta}{\eta^2}\right)^{-\frac{1}{2}}\frac{dy}{y^2} = T\big(\tfrac{1}{x}\big),\ 0<x<1.$$

By using a change of variables $y = xr$, $\eta = xt$, and introducing a new function

$$\varphi(x) := x^{1+\sigma} f\left(\frac{1}{x}\right), \quad 0<x<1, \quad \text{where} \quad \sigma>0,$$

this equation can be written as

$$\frac{1}{\sqrt{2}}\int_0^1 \left(\int_r^1 \frac{\varphi(xt)}{t^{3+\sigma}}\,dt\right)^{-\frac{1}{2}}\frac{dr}{r^2}\,x^{\frac{\sigma}{2}} = T\big(\tfrac{1}{x}\big).$$

Moreover we set

$$\psi(x) = \sqrt{2}\,x^{-\frac{\sigma}{2}}T\big(\tfrac{1}{x}\big).$$

Then the resultant equation becomes

$$\int_0^1 \left(\int_r^1 \frac{\varphi(xt)}{t^{3+\sigma}}\,dt\right)^{-\frac{1}{2}}\frac{dr}{r^2} = \psi(x),\ 0<x<1. \tag{2.1}$$

By defining a mapping $\mathcal{F}$ by

$$\mathcal{F}\varphi(x) := \int_0^1 \left(\int_r^1 \frac{\varphi(xt)}{t^{3+\sigma}}\,dt\right)^{-\frac{1}{2}}\frac{dr}{r^2}, \tag{2.2}$$

where $0<x<1$, equation (2.1) is written simply as

$$\mathcal{F}\varphi = \psi. \tag{2.3}$$

Let $*$ denote a transformation defined by $\phi^*(x) = \phi(\frac{1}{x})$ for a function ϕ, and let $mx^{\ell}\cdot$ denote the multiplication operator by the function mx^{ℓ}. Then the reduction procedure described above is illustrated by the following commutative diagram:

(2.4)

$$\begin{array}{ccc}
f \in \mathcal{C}^{\alpha}(1,\infty)_{1+\sigma} & \xrightarrow{\ \mathcal{B}\ } & \mathcal{C}^{\alpha+\frac{1}{2}}(1,\infty)_{-\frac{\sigma}{2}} \ni T \\
{\scriptstyle *}\downarrow\cong & & {\scriptstyle *}\downarrow\cong \\
f^* \in \mathcal{C}^{\alpha}(0,1)_{-1-\sigma} & \longrightarrow & \mathcal{C}^{\alpha+\frac{1}{2}}(0,1)_{\frac{\sigma}{2}} \ni T^* \\
{\scriptstyle x^{1+\sigma}\cdot}\downarrow\cong & & {\scriptstyle \sqrt{2}\,x^{-\frac{\sigma}{2}}\cdot}\downarrow\cong \\
\varphi \in \mathcal{C}^{\alpha}(0,1)_0 & \xrightarrow{\ \mathcal{F}\ } & \mathcal{C}^{\alpha+\frac{1}{2}}(0,1)_0 \ni \psi.
\end{array}$$

The vertical arrows in the diagram (2.4) are homeomorphisms, which is guaranteed by

Lemma 2.1. *Let $\mathcal{C}^{\alpha}(I)_{\eta}$ be the function space defined by* (1.6) *for each $\alpha \in (0,1]$, $\eta \in \boldsymbol{R}$. Then*:
(1) *The multiplication operator by x^{ℓ} gives a homeomorphism of $\mathcal{C}^{\alpha}(I)_{\eta}$ onto $\mathcal{C}^{\alpha}(I)_{\eta+\ell}$ for each $\ell \in \boldsymbol{R}$.*
(2) *A transformation $* : \phi^*(x) = \phi(\frac{1}{x})$ gives a homeomorphism of $\mathcal{C}^{\alpha}(1,\infty)_{-\eta}$ onto $\mathcal{C}^{\alpha}(0,1)_{\eta}$.*

Proof. Because (1) is direct from the definition (1.6), we shall prove only (2). Let $\phi \in \mathcal{C}^{\alpha}(1,\infty)_{-\eta}$. Then, by a change of variables $x = h^{-1}$, $y = k^{-1}$, we obtain

$$\begin{aligned}
|\phi^*|_{\alpha,\eta} &= \sup_{\substack{0<x,y<1\\ x\neq y}} \frac{|x^{\alpha-\eta}\phi^*(x) - y^{\alpha-\eta}\phi^*(y)|}{|x-y|^{\alpha}} \\
&= \sup_{\substack{1<h,k<\infty\\ h\neq k}} \frac{|k^{\alpha}h^{\eta}\phi(h) - h^{\alpha}k^{\eta}\phi(k)|}{|h-k|^{\alpha}} \\
&\le 2|\phi|_{-\eta} + |\phi|_{\alpha,-\eta},
\end{aligned}$$

because $|h^{\alpha} - k^{\alpha}| \le |h-k|^{\alpha}$ for $1<h,k<\infty$, $0<\alpha\le 1$. This shows that $\phi^* \in \mathcal{C}^{\alpha}(0,1)_{\eta}$ and the correspondence $\phi \mapsto \phi^* : \mathcal{C}^{\alpha}(1,\infty)_{-\eta} \to \mathcal{C}^{\alpha}(0,1)_{\eta}$ is continuous. In a similar way we can show that the inverse $\phi^* \mapsto \phi$ gives a continuous map from $\mathcal{C}^{\alpha}(0,1)_{\eta}$ to $\mathcal{C}^{\alpha}(1,\infty)_{-\eta}$. □

Thus we have:

Proposition 2.2. *There is a commutative diagram* (2.4)*, where the vertical arrows are homeomorphisms.*

Proposition 2.2 tells us that the proof of Theorem 1.2 is reduced to showing that $\mathcal{F}$ defined by (2.2) maps a sufficiently small neighborhood of a positive, constant function c in $\mathcal{C}^{\alpha}(0,1)_0$ homeomorphically onto a neighborhood of $\sqrt{2}\,c'$ in $\mathcal{C}^{\alpha+\frac{1}{2}}(0,1)_0$.

3. Mapping $\mathcal{F}$. In this section we study the mapping $\mathcal{F}$ to establish the following

Proposition 3.1. *Let $0<\alpha<\frac{1}{2}$, $\sigma>0$ and set*

$$U := \{\varphi \in \mathcal{C}^{\alpha}(0,1)_0 : \inf_{0<x<1} \varphi(x) > 0\}.$$

Then $\mathcal{F}$ defined by (2.2) is a C^1-mapping of U to $\mathcal{C}^{\alpha+\frac{1}{2}}(0,1)_0$. The Fréchet derivative $\mathcal{F}'(\varphi_0)$ of $\mathcal{F}$ at a function $\varphi_0 \in U$ is given by

(3.1) $$\mathcal{F}'(\varphi_0)\varphi(x) = -\frac{1}{2}\int_0^1 \frac{\varphi(xt)}{t^{3+\sigma}}\,dt \int_0^t \left(\int_r^1 \frac{\varphi_0(xs)}{s^{3+\sigma}}\,ds\right)^{-\frac{3}{2}} \frac{dr}{r^2},$$

where $0<x<1$. In particular, the Fréchet derivative $\mathcal{F}'(c)$ of $\mathcal{F}$ at a constant function c with $c>0$ is written as

$$\mathcal{F}'(c)\varphi(x) = -\int_0^1 \Phi(t)\varphi(xt)dt, \tag{3.2}$$

where $\Phi(t)$ is a function defined by

$$\Phi(t) := \frac{(2+\sigma)^{\frac{3}{2}}}{2\,c^{\frac{3}{2}}} \frac{1}{t^{3+\sigma}} \int_0^t \frac{s^{1+\frac{3}{2}\sigma}}{(1-s^{2+\sigma})^{\frac{3}{2}}}\,ds. \tag{3.3}$$

The proof of Proposition 3.1 is a combination of four lemmas.

Lemma 3.2. *If $\varphi \in U$ then the function*

$$\psi(x) := (\mathcal{F}\varphi)(x) = \int_0^1 \left(\int_r^1 \frac{\varphi(xt)}{t^{3+\sigma}}\,dt\right)^{-\frac{1}{2}} \frac{dr}{r^2}$$

belongs to $\mathcal{C}^{\alpha+\frac{1}{2}}(0,1)_0$.

Proof. By $\varphi \in U$, $(\int_r^1 \frac{\varphi(xt)}{t^{3+\sigma}}\,dt)^{-\frac{1}{2}} \lesssim \frac{r^{1+\frac{\sigma}{2}}}{\sqrt{1-r}}$. This yields $|\psi|_0 < \infty$. To prove $|\psi|_{\alpha+\frac{1}{2},0} < \infty$, we assume $x>y$ without loss of generality. Since

$$\begin{aligned} &x^{\alpha+\frac{1}{2}}\psi(x) - y^{\alpha+\frac{1}{2}}\psi(y) \\ &\quad = (x^{\alpha+\frac{1}{2}} - y^{\alpha+\frac{1}{2}})\psi(x) + y^{\alpha+\frac{1}{2}}(\psi(x)-\psi(y)) \end{aligned}$$

and $|(x^{\alpha+\frac{1}{2}} - y^{\alpha+\frac{1}{2}})\psi(x)| \lesssim |x-y|^{\alpha+\frac{1}{2}}$, it suffices to show that $|y^{\alpha+\frac{1}{2}}(\psi(x)-\psi(y))| \lesssim |x-y|^{\alpha+\frac{1}{2}}$.

By an elementary calculation with an interchange of the order of integration, we get

$$\begin{aligned} &\psi(x)-\psi(y) \\ &= \int_0^1 d\theta \frac{d}{d\theta}\int_0^1 \left(\int_r^1 \frac{\theta\varphi(x\xi)+(1-\theta)\varphi(y\xi)}{\xi^{3+\sigma}}\,d\xi\right)^{-\frac{1}{2}} \frac{dr}{r^2} \\ &= -\frac{1}{2}\int_0^1 d\theta \int_0^1 \left(\int_r^1 \cdots d\xi\right)^{-\frac{3}{2}} \frac{dr}{r^2} \int_r^1 \frac{\varphi(xt)-\varphi(yt)}{t^{3+\sigma}}\,dt \\ &= -\frac{1}{2}\int_0^1 d\theta \int_0^1 \frac{\varphi(xt)-\varphi(yt)}{t^{3+\sigma}}\,dt \int_0^t \left(\int_r^1 \cdots d\xi\right)^{-\frac{3}{2}} \frac{dr}{r^2}. \end{aligned}$$

Therefore, by putting

(3.4) $$\Phi(x,y;t) := -\frac{1}{2t^{3+\sigma}} \times \int_0^t \frac{dr}{r^2}\int_0^1 \left(\int_r^1 \frac{\theta\varphi(x\xi)+(1-\theta)\varphi(y\xi)}{\xi^{3+\sigma}}\,d\xi\right)^{-\frac{3}{2}} d\theta,$$

we obtain

$$\psi(x)-\psi(y) = \int_0^1 \Phi(x,y;t)(\varphi(xt)-\varphi(yt))dt.$$

This leads to

$$\begin{aligned} &y^{\alpha+\frac{1}{2}}(\psi(x)-\psi(y)) \\ &\quad = y^{\alpha+\frac{1}{2}}\int_0^1 \Phi(x,y;t)(\varphi(xt)-\varphi(y))dt \\ &\qquad - y^{\alpha+\frac{1}{2}}\int_0^1 \Phi(x,y;t)(\varphi(yt)-\varphi(y))dt \\ &\quad = I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &:= y^{\alpha+\frac{1}{2}}\int_y^x \Phi\left(x,y;\frac{s}{x}\right)(\varphi(s)-\varphi(y))\,\frac{ds}{x}, \\ I_2 &:= y^{\alpha+\frac{1}{2}}\int_0^y \left(\frac{1}{x}\,\Phi\left(x,y;\frac{s}{x}\right) - \frac{1}{y}\,\Phi\left(x,y;\frac{s}{y}\right)\right) (\varphi(s)-\varphi(y))ds. \end{aligned}$$

Since, in (3.4), $\theta\varphi(x\xi)+(1-\theta)\varphi(y\xi) \geq \inf\varphi > 0$, $\Phi(x,y;t)$ satisfies

$$|\Phi(x,y;t)| \lesssim \frac{1}{t^{3+\sigma}}\int_0^t \frac{r^{1+\frac{3}{2}\sigma}}{(1-r^{2+\sigma})^{\frac{3}{2}}}\,dr \lesssim t^{\frac{\sigma}{2}-1}(1-t)^{-\frac{1}{2}}. \tag{3.5}$$

Moreover it follows from (3.4) that the derivative $\Phi'(x,y;t)$ of $\Phi(x,y;t)$ with respect to t satisfies

$$|t(1-t)\Phi'(x,y;t)| \lesssim t^{\frac{\sigma}{2}-1}(1-t)^{-\frac{1}{2}}. \tag{3.6}$$

Because of $\varphi \in \mathcal{C}^{\alpha}(0,1)_0$, φ satisfies

$$|\varphi(s)-\varphi(y)| \lesssim \left(1-\frac{y}{s}\right)^{\alpha}, \quad y \leq s. \tag{3.7}$$

Hence, by (3.5) and a substitution $s = y+\eta(x-y)$, we have

$$\begin{aligned} |I_1| &\lesssim y^{\alpha+\frac{1}{2}}\int_y^x \left(\frac{s}{x}\right)^{\frac{\sigma}{2}-1}\left(1-\frac{s}{x}\right)^{-\frac{1}{2}}\left(1-\frac{y}{s}\right)^{\alpha}\frac{ds}{x} \\ &= (x-y)^{\alpha+\frac{1}{2}}\left(\frac{y}{x}\right)^{\alpha+\frac{1}{2}} \\ &\quad \times \int_0^1 \left(\frac{y+\eta(x-y)}{x}\right)^{\frac{\sigma}{2}-\alpha-1}\frac{\eta^{\alpha}}{(1-\eta)^{\frac{1}{2}}}\,d\eta. \end{aligned}$$

Because of $x\eta \le y + \eta(x-y) \le x$, $y/x \le 1$, we get $I_1 \lesssim (x-y)^{\alpha+\frac{1}{2}}$. Moreover, I_2 is rewritten as

$$I_2 = y^{\alpha+\frac{1}{2}} \int_0^1 \int_{y/x}^1 \frac{d}{d\eta}\,(\eta\,\Phi(x,y,\eta t))d\eta\,(\varphi(y)-\varphi(yt))dt,$$

which can be evaluated by using (3.5), (3.6), (3.7) and the substitution $t = (1-s)/(1-\eta s)$ so that

$$\begin{aligned}|I_2| &\lesssim y^{\alpha+\frac{1}{2}} \int_0^1 \int_{y/x}^1 (\eta t)^{\frac{\sigma}{2}-1}(1-\eta t)^{-\frac{3}{2}}d\eta\,(1-t)^{\alpha}dt \\ &= y^{\alpha+\frac{1}{2}} \int_{y/x}^1 \eta^{\frac{\sigma}{2}-1}d\eta \int_0^1 t^{\frac{\sigma}{2}-1}(1-\eta t)^{-\frac{3}{2}}(1-t)^{\alpha}dt \\ &= y^{\alpha+\frac{1}{2}} \int_{y/x}^1 \eta^{\frac{\sigma}{2}-1}(1-\eta)^{\alpha-\frac{1}{2}}\,d\eta \int_0^1 \frac{s^{\alpha}(1-s)^{\frac{\sigma}{2}-1}}{(1-\eta s)^{\frac{\sigma}{2}+\alpha-\frac{1}{2}}}\,ds.\end{aligned}$$

Taking the assumption $\alpha < \frac{1}{2}$ into account, we get

$$\begin{aligned}|I_2| &\lesssim y^{\alpha+\frac{1}{2}} \int_{y/x}^1 \eta^{\frac{\sigma}{2}-1}(1-\eta)^{\alpha-\frac{1}{2}}\,d\eta \\ &\lesssim y^{\alpha+\frac{1}{2}}\left(1-\frac{y}{x}\right)^{\alpha+\frac{1}{2}} \le (x-y)^{\alpha+\frac{1}{2}}.\end{aligned}$$

Thus $|\psi|_{\alpha+\frac{1}{2},0} < \infty$, and so, $\psi \in \mathcal{C}^{\alpha+\frac{1}{2}}(0,1)_0$. □

In order to show that $\mathcal{F}$ is Fréchet differentiable, the following generalization of Lemma 3.2 is useful.

Lemma 3.3. *Let $\phi_0 \in U$, $\phi_1, \phi_2 \in \mathcal{C}^{\alpha}(0,1)_0$, and let $\chi(x)$ be a function defined by*

$$\chi(x) = \int_0^1 \left(\int_r^1 \frac{\phi_0(xs)}{s^{3+\sigma}}\,ds\right)^{-\frac{5}{2}} \prod_{i=1}^{2}\left(\int_r^1 \frac{\phi_i(xt)}{t^{3+\sigma}}\,dt\right)\frac{dr}{r^2}.$$

Then χ belongs to $\mathcal{C}^{\alpha+\frac{1}{2}}(0,1)_0$ with the norm

$$\|\chi\|_{\alpha+\frac{1}{2},0} \lesssim \|\phi_0\|_{\alpha,0}\,\|\phi_1\|_{\alpha,0}\,\|\phi_2\|_{\alpha,0}.$$

Proof. This lemma can be proved by the same method as in the proof of Lemma 3.2. □

Lemma 3.4. *Let $\varphi_0 \in U$ and let $\mathcal{F}'(\varphi_0)$ be an operator defined by* (3.1)*. Then*:

(1) *For each $\varphi \in \mathcal{C}^{\alpha}(0,1)_0$,*

$$\lim_{\theta\to 0} \frac{\mathcal{F}(\varphi_0+\theta\varphi)-\mathcal{F}\varphi_0}{\theta} = \mathcal{F}'(\varphi_0)\varphi$$

in the norm of $\mathcal{C}^{\alpha+\frac{1}{2}}(0,1)_0$.

(2) *For $\varphi_1 \in U$ near φ_0, the operator norm of $\mathcal{F}'(\varphi_1)-\mathcal{F}'(\varphi_0)$ is evaluated as*

$$\|\mathcal{F}'(\varphi_1)-\mathcal{F}'(\varphi_0)\| \lesssim \|\varphi_1-\varphi_0\|_{\alpha,0}.$$

In particular, $\mathcal{F}'(\varphi_0)$ is continuous in φ_0 in the sense of the operator norm.

Proof. Let θ be so small that $\varphi_0+\theta\varphi \in U$ and let $\Delta_i(x,\theta)$, $i=0,1,2$, be functions defined by

$$\int_0^1 \left(\int_r^1 \frac{(\varphi_0+\theta\varphi)(xs)}{s^{3+\sigma}}\,ds\right)^{-\frac{1}{2}-i}\left(\int_r^1 \frac{\varphi(xt)}{t^{3+\sigma}}\,dt\right)^{i}\frac{dr}{r^2}.$$

Then, by (2.2), for each $x \in (0,1)$,

$$\begin{aligned}&\mathcal{F}(\varphi_0+\theta\varphi)(x)-\mathcal{F}\varphi_0(x) \\ &\quad= \int_0^{\theta} \frac{d}{d\tau}\,\Delta_0(x,\tau)d\tau = -\frac{1}{2}\int_0^{\theta} \Delta_1(x,\tau)d\tau.\end{aligned}$$

On the other hand, if we define $\mathcal{F}'(\varphi_0)$ by (3.1) then, by an interchange of the order of integration,

$$\mathcal{F}'(\varphi_0)\varphi(x) = -\frac{1}{2}\,\Delta_1(x,0). \tag{3.8}$$

Hence

$$\begin{aligned}&\frac{\mathcal{F}(\varphi_0+\theta\varphi)(x)-\mathcal{F}\varphi_0(x)}{\theta} - \mathcal{F}'(\varphi_0)\varphi(x) \\ &\quad= -\frac{1}{2\theta}\int_0^{\theta} d\tau \int_0^{\tau} \frac{d}{d\omega}\,\Delta_1(x,\omega)d\omega \\ &\quad= \frac{3}{4\theta}\int_0^{\theta} d\tau \int_0^{\tau} \Delta_2(x,\omega)d\omega.\end{aligned}$$

In view of Lemma 3.3, the norm of $\Delta_2(\cdot,\omega)$ in $\mathcal{C}^{\alpha+\frac{1}{2}}(0,1)_0$ is bounded uniformly with respect to ω near 0. This proves (1).

By (3.8), we have, for $\varphi_0, \varphi_1 \in U$, $\varphi \in \mathcal{C}^{\alpha}(0,1)_0$,

$$\begin{aligned}&\mathcal{F}'(\varphi_1)\varphi(x)-\mathcal{F}'(\varphi_0)\varphi(x) \\ &\quad= \frac{3}{4}\int_0^1 d\theta \int_0^1 \left(\int_r^1 \frac{((1-\theta)\varphi_0+\theta\varphi_1)(xs)}{s^{3+\sigma}}\,ds\right)^{-\frac{5}{2}} \\ &\qquad \left(\int_r^1 \frac{(\varphi_1-\varphi_0)(xs)}{s^{3+\sigma}}\,ds\right)\left(\int_r^1 \frac{\varphi(xt)}{t^{3+\sigma}}\,dt\right)\frac{dr}{r^2}.\end{aligned}$$

This, combined with Lemma 3.3, proves (2). □

Lemma 3.5. *The mapping $\mathcal{F}$ is Fréchet differentiable. The Fréchet derivative $\mathcal{F}'(\varphi_0)$, which is given by* (3.1)*, is continuous in φ_0.*

Proof. We prove the lemma by a standard discussion (see, e.g., [4, Lemma 1.15]). Lemma 3.4(1) implies that, for small φ and $\theta \in [0,1]$,

$$\frac{d}{d\theta}\,\mathcal{F}(\varphi_0+\theta\varphi) = \mathcal{F}'(\varphi_0+\theta\varphi)\varphi$$

in the space $\mathcal{C}^{\alpha+\frac{1}{2}}(0,1)_0$. This leads to

$$\begin{aligned}&\mathcal{F}(\varphi_0+\varphi)-\mathcal{F}(\varphi_0) \\ &\quad= \int_0^1 \mathcal{F}'(\varphi_0+\theta\varphi)\varphi\,d\theta \\ &\quad= \mathcal{F}'(\varphi_0)\varphi + \int_0^1 (\mathcal{F}'(\varphi_0+\theta\varphi)-\mathcal{F}'(\varphi_0))\varphi\,d\theta\end{aligned}$$

for small $\varphi \in \mathcal{C}^{\alpha}(0,1)_0$. This, together with Lemma 3.4(2), proves the lemma. □

4. Proof of the main theorem. We first prove a proposition that is crucial for the proof of Theorem 1.2.

Proposition 4.1. *Let* $0 < \alpha < \frac{1}{2}$, $\sigma > 0$. *Then the operator* $\mathcal{F}'(c)$ *given by* (3.2) *is a homeomorphism of* $\mathcal{C}^{\alpha}(0,1)_0$ *onto* $\mathcal{C}^{\alpha+\frac{1}{2}}(0,1)_0$.

Proof. We define an operator J_Φ by

$$J_\Phi\varphi(x) = \int_0^1 \Phi(t)\varphi(xt)dt. \tag{4.1}$$

Then $\mathcal{F}'(c) = -J_\Phi$.

The operator J_Φ of the form (4.1) is a multiplicative Wiener-Hopf integral operator. The reason for the use of this terminology and a general theory of the operator may be found in Iwasaki and Kamimura [2, p. 115] and [1]. We here use a result for a singular multiplicative Wiener-Hopf integral operator:

Lemma 4.2 (Theorem B in [1]). *Let*

$$\Phi(t) = At^{\epsilon-1}(1-t^\beta)^{\delta-1} + R(t),\ \beta,\epsilon > 0,\ 0 < \delta < 1$$

with $A \neq 0$ *satisfy*

$$R(t) \in C(0,1] \cap C^2(0,1), \quad |R(t)| \lesssim t^{\nu-1},$$
$$|R'(t)| \lesssim t^{\nu-2}(1-t)^{\rho-1}, \quad |R''(t)| \lesssim t^{\nu-3}(1-t)^{\rho-2},$$

with $\nu,\rho > 0$, *and let* $0 < \alpha < 1-\delta$. *Then* J_Φ, *which is a bounded linear operator from* $\mathcal{C}^\alpha(0,1)_0$ *to* $\mathcal{C}^{\alpha+\delta}(0,1)_0$, *is a homeomorphism of* $\mathcal{C}^\alpha(0,1)_0$ *onto* $\mathcal{C}^{\alpha+\delta}(0,1)_0$ *if and only if*

$$\int_0^1 \Phi(t)t^z dt \neq 0, \quad \operatorname{Re} z \geq 0. \tag{4.2}$$

Let us verify that Φ defined by (3.3) satisfies conditions in the lemma. By means of the hypergeometric function $F(\alpha,\beta,\gamma;\cdot)$, we can compute

$$\int_0^t \frac{s^{1+\frac{3}{2}\sigma}}{(1-s^{2+\sigma})^{\frac{3}{2}}}ds = \frac{2t^{\frac{\sigma}{2}}}{2+\sigma}\left\{\left(\frac{1}{\sqrt{1-t^{2+\sigma}}}-1\right) + \left(1 - F\left(\frac{1}{2}, \frac{\sigma}{4+2\sigma}, \frac{4+3\sigma}{4+2\sigma}; t^{2+\sigma}\right)\right)\right\}.$$

Therefore the function $\Phi(t)$ in (3.3) is expressed as

$$\Phi(t) = A\frac{t^{\frac{\sigma}{2}-1}}{\sqrt{1-t^{2+\sigma}}} + R(t), \quad A := \frac{\sqrt{2+\sigma}}{c^{\frac{3}{2}}},$$

in terms of a function $R(t)$ with

$$R(t) \in C(0,1] \cap C^2(0,1), \quad |R(t)| \lesssim t^{\frac{\sigma}{2}-1},$$
$$|R'(t)| \lesssim t^{\frac{\sigma}{2}-2}(1-t)^{-\frac{1}{2}}, \quad |R''(t)| \lesssim t^{\frac{\sigma}{2}-3}(1-t)^{-\frac{3}{2}}.$$

To prove that $\Phi(t)$ in (3.3) satisfies the condition (4.2), we employ the following (see [1, Lemma 1.9]): *If* $\Phi(t) \in L^1(0,1) \cap C^1(0,1)$ *satisfies*

$$\Phi(t),\ (t\Phi(t))' \geq 0, \quad t \in (0,1), \quad \Phi(t) \not\equiv 0, \tag{4.3}$$

then (4.2) *is fulfilled.*

In what follows, we shall show that $(t\Phi(t))' \geq 0$ for $t \in (0,1)$. By the definition (3.3) and an elementary computation, we have

$$\frac{2c^{\frac{3}{2}}}{(2+\sigma)^{\frac{3}{2}}}(t\Phi(t))' = \left(\frac{1}{t^{2+\sigma}}\int_0^t \frac{s^{1+\frac{3}{2}\sigma}}{(1-s^{2+\sigma})^{\frac{3}{2}}}ds\right)'$$
$$= \left(\frac{1}{t^{2+\sigma}}\int_0^t \frac{s^{1+\frac{3}{2}\sigma}}{(1-s^{2+\sigma})^{\frac{3}{2}}}ds\right)'$$
$$= \frac{2+\sigma}{t^{3+\sigma}}\int_0^t\left(\frac{1}{(1-t^{2+\sigma})^{\frac{3}{2}}} - \frac{1}{(1-s^{2+\sigma})^{\frac{3}{2}}}\right)s^{1+\frac{3}{2}\sigma}ds + \frac{\sigma}{4+3\sigma}\frac{t^{\frac{\sigma}{2}-1}}{(1-t^{2+\sigma})^{\frac{3}{2}}}.$$

Since, for $0 < s < t$, $\frac{1}{(1-t^{2+\sigma})^{\frac{3}{2}}} - \frac{1}{(1-s^{2+\sigma})^{\frac{3}{2}}} > 0$, we get $(t\Phi(t))' > 0$ for $t \in (0,1)$. Thus Φ defined by (3.3) satisfies (4.3), and so (4.2). □

Proof of Theorem 1.2. By Propositions 3.1, 4.1 we can apply the implicit function theorem (see, e.g., [4, Theorem 1.20]) to conclude that $\mathcal{F}$ maps a sufficiently small neighborhood of a positive, constant function c in $\mathcal{C}^\alpha(0,1)_0$ homeomorphically onto a neighborhood of $\sqrt{2}\,c'$ in $\mathcal{C}^{\alpha+\frac{1}{2}}(0,1)_0$. This, together with Proposition 2.2, proves Theorem 1.2. □

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