A note on Hayman's problem and the sharing value

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Abstract: Let f be a nonconstant meromorphic functions, n, k be two positive integers. Suppose that f^n and $(f^n)^{(k)}$ share the value $a(\neq 0,\infty)$ CM. If either (1) $n > k+2$, or (2) $n > k+1$ and $\bar{N}(r,f) = \lambda T(r,f)(\lambda \in [0,\frac{1}{2}))$, then $f^n = (f^n)^{(k)}$ and f assumes the form

$$f(\mathrm{z}) = \mathrm{c}e^{\frac{\lambda}{n}z}$$

where c is a nonzero constant and $\lambda^k = 1$.

Key words: Meromorphic functions; uniqueness theorems; shared value.

1. Introduction and main results. In this paper, a meromorphic function will mean meromorphic in the whole complex plane. We assume that the reader is familiar with the standard notations of the Nevanlinna theory such as $T(r,f), N(r,f), m(r,f)$ ([1,2]).

For any nonconstant meromorphic function f, we denote by $S(r,f)$ any quantity satisfying

$$S(r,f) = \circ\{T(r,f)\}, r \to \infty$$

possibly outside of a set of finite linear measure. Let $f(z)$ and $g(z)$ be two meromorphic functions, and let a be a finite complex number. If $f(z) - a$ and $g(z) - a$ assume the same zeros with the same multiplicities, then we say that f and g share the value a CM (counting multiplicities) (see [2] pp. 115–116).

In 1959, W. K. Hayman [3] proposed the following conjecture and until 1995 it was proved by W. Bergweiler and A. Eremenko [4], H. H. Chen and M. L. Fang [5] separately.

Theorem A. *If f is a transcendental meromorphic function, then $f^n f'$ assumes every finite non-zero complex value infinitely often for any positive integer n.*

In 1998, Y. F. Wang and M. L. Fang [6] proved the following result.

Theorem B. *If f is a transcendental meromorphic function, n, k be two positive integers and $n \geq k+1$, then $(f^n)^{(k)}$ assumes every finite non-zero complex value infinitely often.*

The uniqueness theory of entire and meromorphic functions has grown up to an extensive subfield of the value distribution theory. In particular, the subtopic that a meromorphic function f and its derivative $f^{(k)}$ share one finite non-zero value a CM is well investigated (see [7–12]).

Theorem C ([7]). *Let f be a nonconstant entire function and $k, n(\geq k+1)$ be two positive integers. If f^n and $(f^n)^{(k)}$ share 1 CM, then $f^n = (f^n)^{(k)}$ and f assumes the form*

$$f(\mathrm{z}) = \mathrm{c}e^{\frac{\lambda}{n}z}$$

where c is a nonzero constant and $\lambda^k = 1$.

Theorem D ([8, Theorem 1]). *Let f be a nonconstant meromorphic function and $n \geq 4$ be a positive integer. If f^n and $(f^n)'$ share 1 CM, then $f^n = (f^n)'$ and f assumes the form*

$$f(\mathrm{z}) = \mathrm{c}e^{\frac{1}{n}z}$$

where c is a nonzero constant.

Theorem E ([8, Theorem 2]). *Let f be a nonconstant meromorphic function and $n(\geq k+5), k$ be two positive integers. If f^n and $(f^n)^{(k)}$ share 1 CM, then $f^n = (f^n)^{(k)}$ and f assumes the form*

$$f(\mathrm{z}) = \mathrm{c}e^{\frac{\lambda}{n}z}$$

where c is a nonzero constant and $\lambda^k = 1$.

Theorem F ([11, Theorem 1.2]). *Let f be a nonconstant meromorphic function and $n(> k+1+\sqrt{k+1}), k$ be two positive integers. If f^n and $(f^n)^{(k)}$ share 1 CM, then $f^n = (f^n)^{(k)}$ and f assumes the form*

$$f(\mathrm{z}) = \mathrm{c}e^{\frac{\lambda}{n}z}$$

where c is a nonzero constant and $\lambda^k = 1$.

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J. Zhang and L. Yang [11] asked a question: Can n in Theorem E be reduced? Recently, S. Li and Z. Gao [12, Theorem 1.1] answered this question in the case of $\bar{N}(r,f)=S(r,f)$, they proved the following theorem.

Theorem G. *Let f be a nonconstant meromorphic function, such that $\bar{N}(r,f)=S(r,f)$. Suppose that f^n and $(f^n)'$ share 1 CM. If either* (1) $n\geq 3$, *or* (2) $n=2$ *and* $\bar{N}(r,\frac{1}{f})=O(N_{(3}(r,\frac{1}{f}))$, *then* $f^n=(f^n)'$ *and f assumes the form*

$$f(z)=ce^{\frac{1}{n}z}$$

where c is a nonzero constant.

It is thus natural to ask whether the conditions in Theorem D and Theorem G holds for the k_{th} derivative, namely, Can n in Theorem E and Theorem F be reduced? In this paper we investigate this problem and prove the following result.

Theorem 1. *Let f be a nonconstant meromorphic functions, n,k be two positive integers. Suppose that f^n and $(f^n)^{(k)}$ share the value $a(\neq 0,\infty)$ CM. If either* (1) $n\geq k+2$, *or* (2) $n\geq k+1$ *and* $\bar{N}(r,f)=\lambda T(r,f)(\lambda\in[0,\frac{1}{2}))$, *then* $f^n=(f^n)^{(k)}$ *and f assumes the form*

$$f(z)=ce^{\frac{\lambda}{n}z}$$

where c is a nonzero constant and $\lambda^k=1$.

2. Some lemmas. To prove our results, we need some preliminary results.

Lemma 1 ([7, Lemma 3]). *Let f be a nonconstant meromorphic function and $n(\geq k+2),k$ be two positive integers. If f^n and $(f^n)^{(k)}$ share the value $a(\neq 0,\infty)$ CM, then one of the following two cases must occur*:

(1) $f^n=(f^n)^{(k)}$;

(2) $N(r,\frac{1}{f})\leq\frac{1}{n-k-1}\bar{N}(r,f)+S(r,f)$.

Lemma 2 ([10, Lemma 2.10]). *Let f be a nonconstant meromorphic function and $n(\geq k+2)$, k be two positive integers. If $f^n=(f^n)^{(k)}$, then f assumes the form*

$$f(z)=ce^{\frac{\lambda}{n}z}$$

where c is a nonzero constant and $\lambda^k=1$.

Lemma 3 ([1, Theorem 3.1]). *Let f be a nonconstant meromorphic function in the complex plane and k be a positive integer. If $f^n=(f^n)^{(k)}$, Then*

$$m\left(r,\frac{f^{(k)}}{f}\right)=S(r,f)$$

3. Proof of Theorem.

3.1. Proof of Theorem 1. Suppose $a=1$ (the general case following by considering $\frac{f^n}{a}$ instead of f^n) and $f^n\not\equiv(f^n)^{(k)}$. We set

$$F=\frac{1}{f^n}\left(\frac{(f^n)^{(k+1)}}{(f^n)^{(k)}-1}-\frac{(f^n)'}{f^n-1}\right). \tag{3.1}$$

From the fundamental estimate of logarithmic derivative it follows that

$$\begin{aligned} m(r,F)&\leq m\left(r,\frac{(f^n)^{(k+1)}}{f^n((f^n)^{(k)}-1)}\right)\\ &\quad+m\left(r,\frac{(f^n)'}{f^n(f^n-1)}\right)\\ &=m\left(r,\frac{(f^n)^{(k+1)}}{(f^n)^{(k)}((f^n)^{(k)}-1)}\frac{(f^n)^{(k)}}{f^n}\right)\\ &\quad+m\left(r,\frac{(f^n)'}{f^n(f^n-1)}\right)\\ &\leq m\left(r,\left(\frac{(f^n)^{(k+1)}}{(f^n)^{(k)}-1}-\frac{(f^n)^{(k+1)}}{(f^n)^{(k)}}\right)\frac{(f^n)^{(k)}}{f^n}\right)\\ &\quad+m\left(r,\frac{(f^n)'}{(f^n)-1}-\frac{(f^n)'}{f^n}\right)\\ &\leq m\left(r,\frac{(f^n)^{(k+1)}}{(f^n)^{(k)}-1}\right)+m\left(r,\frac{(f^n)^{(k+1)}}{(f^n)^{(k)}}\right)\\ &\quad+m\left(r,\frac{(f^n)^{(k)}}{f^n}\right)+m\left(r,\frac{(f^n)'}{(f^n)-1}\right)\\ &\quad+m\left(r,\frac{(f^n)'}{f^n}\right)\\ &\leq S(r,f). \end{aligned} \tag{3.2}$$

From (3.1), if z_0 is a pole of f with multiplicity $\geq m$, then z_0 is a zero of F with multiplicity at least $nm-1$, i.e.,

$$F(z)=O((z-z_0)^{nm-1}). \tag{3.3}$$

We consider the following two cases:

Case 1. $F^2-F'\equiv 0$. Solving this equation, we have

$$F(z)=\frac{1}{c-z} \tag{3.4}$$

where c is a constant. Substituting (3.4) into (3.1) gives

$$\frac{1}{c-z}=\frac{1}{f^n}\left(\frac{(f^n)^{(k+1)}}{(f^n)^{(k)}-1}-\frac{(f^n)'}{f^n-1}\right). \tag{3.5}$$

From (3.5), it is easy to deduce that $f(z)$ is a entire function.

From Theorem C, we get that

$$f^n \equiv (f^n)^{(k)}.$$

This is a contradiction.

Case 2. $F^2 - F' \not\equiv 0$. Since $m(r,F) = S(r,f)$, so $m(r,F') \le m(r,F) + m(r,\frac{F'}{F}) = S(r,f)$.

From (3.3), we deduce that

$$(3.6)\quad N(r,f^n) - 2\bar{N}(r,f) \le N\left(r,\frac{1}{F^2-F'}\right) \le T(r,F^2-F') - m\left(r,\frac{1}{F^2-F'}\right) + O(1) \le N(r,F^2-F') - m\left(r,\frac{1}{F^2-F'}\right) + S(r,f).$$

Since f^n and $(f^n)^{(k)}$ share 1 CM, so

$$(3.7)\quad \frac{(f^n)^{(k)}-1}{f^n-1} = \frac{1}{g^k}$$

where $g(z)(\not\equiv 0)$ is a entire function. It is easy to see that all of zeros of $g(z)$ are poles of $f(z)$ and are simple. Substituting this into (3.1), we get

$$(3.8)\quad F = \frac{1}{f^n}\frac{-kg'}{g}.$$

From (3.8), we can get that the poles of $F^2 - F'$ can only occur at the zeros of f. However, from (3.1), we can deduce that the zeros of f with multiplicity m are all poles of $F^2 - F'$ with multiplicity $2(k+1)$, at most, thus

$$(3.9)\quad N(r,F^2-F') \le 2(k+1)\bar{N}\left(r,\frac{1}{f}\right) \le \frac{2(k+1)}{n}N\left(r,\frac{1}{f^n}\right).$$

From (3.8), we get $F' = \frac{nf'}{f^{n+1}}\frac{kg'}{g} + \frac{1}{f^n}(\frac{-kg'}{g})'$. It follows that

$$F^2 - F' = \frac{1}{f^{2n}}\left\{k^2\left(\frac{g'}{g}\right)^2 - kf^n\left[n\frac{f'}{f}\frac{g'}{g} - \left(\frac{g'}{g}\right)'\right]\right\}$$

$$F^2 - F' = \frac{1}{f^{2n}}\left\{k^2\left(\frac{g'}{g}\right)^2 - kf^n\left[n\frac{f'}{f}\frac{g'}{g} - \frac{(\frac{g'}{g})'}{\frac{g'}{g}}\frac{g'}{g}\right]\right\},$$

i.e.,

$$f^{2n} = \frac{1}{F^2-F'}\left\{k^2\left(\frac{g'}{g}\right)^2 - kf^n\left[n\frac{f'}{f}\frac{g'}{g} - \frac{(\frac{g'}{g})'}{\frac{g'}{g}}\frac{g'}{g}\right]\right\}.$$

It follows that

$$(3.10)\quad 2m(r,f^n) \le m\left(r,\frac{1}{F^2-F'}\right) + m(r,f^n) + S(r,f).$$

From (3.6), (3.9), (3.10) and Lemma 1, we get

$$(3.11)\quad \begin{aligned} T(r,f^n) &= m(r,f^n) + N(r,f^n) \\ &\le m\left(r,\frac{1}{F^2-F'}\right) + 2\bar{N}(r,f) + N(r,F^2-F') - m\left(r,\frac{1}{F^2-F'}\right) + S(r,f) \\ &= 2\bar{N}(r,f) + N(r,F^2-F') + S(r,f) \\ &\le 2\bar{N}(r,f) + \frac{2(k+1)}{n}N\left(r,\frac{1}{f^n}\right) + S(r,f) \\ &\le 2\bar{N}(r,f) + \frac{2(k+1)}{n-k-1}\bar{N}(r,f) + S(r,f) \\ &\le \frac{2n}{n-k-1}\bar{N}(r,f) + S(r,f), \end{aligned}$$

so

$$(3.12)\quad T(r,f) \le \frac{2}{n-k-1}\bar{N}(r,f) + S(r,f) \le 2\bar{N}(r,f) + S(r,f).$$

Case 2.1. $n > k+1$ and $\bar{N}(r,f) = \lambda T(r,f)(\lambda \in [0,\frac{1}{2}))$. By (3.12), we get

$$(1-2\lambda)T(r,f) \le S(r,f)$$

which contradicts the fact that f is nonconstant function.

Case 2.2. $n > k+2$.

It follows from (3.1) and (3.8) that the poles of F can only occur at the zeros of f. If z_0 is a zero of f with multiplicity l, then z_0 is a pole of F with multiplicity at most $k+1$, so

$$(3.13)\quad N(r,F) \le (k+1)\bar{N}\left(r,\frac{1}{f}\right) \le nN\left(r,\frac{1}{f}\right) = N\left(r,\frac{1}{f^n}\right).$$

Suppose that z_0 is a poles of f with multiplicity m. By (3.1), we deduce that z_0 is a zero of F with multiplicity at least $nm-1$. From Lemma 1 and (3.2), we get

$$\bar{N}(r,f) \le \frac{1}{n-1}N\left(r,\frac{1}{F}\right) \le \frac{1}{n-1}T(r,F) + O(1) \le \frac{1}{n-1}N(r,F) + S(r,f)$$

$$
\begin{aligned}
&\le \frac{1}{n-1} N\left(r, \frac{1}{f^n}\right) + S(r,f) \\
&\le \frac{n}{n-1} \frac{1}{n-k-1} \bar{N}(r,f) + S(r,f) \\
&\le \frac{n}{2(n-1)} \bar{N}(r,f) + S(r,f) \\
&= \left(\frac{1}{2} + \frac{1}{2(n-1)}\right) \bar{N}(r,f) + S(r,f) \\
&\le \left(\frac{1}{2} + \frac{1}{2(k+2)}\right) \bar{N}(r,f) + S(r,f) \\
&\le \frac{2}{3} \bar{N}(r,f) + S(r,f)
\end{aligned}
$$

which implies that $\bar{N}(r,f) = S(r,f)$.

By (3.12), we get

$$T(r,f) \le S(r,f)$$

which contradicts the fact that f is nonconstant function.

Thus $f^n \equiv (f^n)^{(k)}$, from Lemma 2, we can get Theorem 1.

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