A_∞ constants between *BMO* and weighted *BMO*

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Abstract: In this short article, we consider estimates of the ratio

$$\|f\|_{BMO(w)}/\|f\|_{BMO}$$

from above and below, where w belongs to Muckenhoupt class A_∞. The upper bound of the ratio was proved by Hytönen and Pérez in [6] with the optimal power. We establish the lower bound of the ratio and give two other proofs of the upper bound.

Key words: BMO; Muckenhoupt classes.

1. Introduction. In this paper, we are interested in estimates of the ratio

$$\|f\|_{BMO(w)}/\|f\|_{BMO}$$

with respect to the weight w belonging to Muckenhoupt class A_∞. Our purposes are to establish the lower bound of the ratio and to give two other proofs of the upper bound due to Hytönen and Pérez in [6].

In [9], Muckenhoupt and Wheeden proved that for any $w \in A_\infty$, it holds $BMO(w) = BMO$. Recently, Hytönen and Pérez [6] gave the upper bound of the ratio;

$$\|f\|_{BMO(w)} \le c_n \|w\|_{A_\infty} \|f\|_{BMO}, \tag{1.1}$$

where $\|w\|_{A_\infty}$ is Wilson's A_∞ constant, see Definition 2.6. Moreover, they [6] proved that the power 1 of $\|w\|_{A_\infty}$ cannot be replaced by any smaller quantity. Main result in this paper is the following lower bound of the ratio.

Theorem 1.1. *There exists $c_n > 0$ such that for any $w \in A_\infty$,*

$$\|f\|_{BMO} \le c_n \log(2[w]_{A_\infty}) \|f\|_{BMO(w)}. \tag{1.2}$$

Remark 1.2.

(a) We do not know whether the order $\log(2[w]_{A_\infty})$ is optimal or not.

(b) If the inequality

$$\|f\|_{BMO} \le c_n \|f\|_{BMO(w)}$$

is true, the exponent 0 of $[w]_{A_\infty}$ is optimal. In fact, for $w(x) = t\chi_E(x) + \chi_{E^c}(x) \in A_1$ with a compact set $E \subset \mathbf{R}^n$ and large t, it follows

$$\|\log w\|_{BMO} = \|\log w\|_{BMO(w)} = \frac{1}{2}\log t.$$

We will give two other proofs of the upper bound (1.1). To verify (1.1) in [6], they used the reverse Hölder inequality;

$$\langle w^{r_w}\rangle_Q^{1/r_w} \le 2\langle w\rangle_Q,$$

for a cube $Q \subset \mathbf{R}^n$ and $r_w = 1 + (c_n\|w\|_{A_\infty})^{-1}$. Our proofs of (1.1) are not based on this type inequality. Our main tools are a dual inequality with the sharp maximal operator $M_\lambda^\sharp$ due to Lerner [7] and another representation of $\|w\|_{A_\infty}$.

These estimates are related to the sharp weighted inequalities for Calderón-Zygmund operators. The sharp weighted inequality for an operator T means the inequality

$$\|Tf\|_{L^p(w)} \le c_{n,p,T}\Phi([w]_{A_p})\|f\|_{L^p(w)} \tag{1.3}$$

with the optimal growth function Φ on $[1,\infty)$ in the sense that Φ cannot be replaced by any smaller function. Recently, Hytönen [5] solved so-called A_2 conjecture i.e., for any Calderón-Zygmund operator T (1.3) holds with $\Phi(t) = t$. By combining this with the extrapolation theorem in [1], we can see that for $p \in (1,\infty)$ (1.3) with $\Phi(t) = t^{\max(1,1/(p-1))}$ holds and the exponent $\max(1, 1/(p-1))$ is optimal. From the upper bound (1.1), it immediately follows

$$\|Tf\|_{BMO(w)} \le c_n\|T\|_{L^\infty\to BMO}\|w\|_{A_\infty}\|f\|_{L^\infty(w)}$$

which corresponds to (1.3) with $p = \infty$. Further,

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they [6] showed the optimality of the exponent 1 of $\|w\|_{A_\infty}$. On the other hand, our lower bound (1.2) yields that

$$\|T\|_{BMO(w)\to BMO(w)} \leq c_n\|T\|_{BMO\to BMO}\|w\|_{A_\infty}\log(2[w]_{A_\infty}).$$

2. Preliminaries. We say w a *weight* if w is a non-negative and locally integrable function. For a subset $E\subset \mathbf{R}^n$, χ_E means the characteristic function of E and $|E|$ denotes the volume of E. By a "cube" Q we mean a cube in $\mathbf{R}^n$ with sides parallel to the coordinate axes. Throughout this article we use the following notations; $w(Q)=\int_Q w dx$, $\langle f\rangle_Q = \frac{1}{|Q|}\int_Q f dx$ and $\langle f\rangle_{Q;w}=\frac{1}{w(Q)}\int_Q fw dx$.

Firstly, we recall definitions of Muckenhoupt classes A_p and BMO spaces.

Definition 2.1. A weight w is said to be in the Muckenhoupt class if the following A_p constant $[w]_{A_p}$ is finite;

$$[w]_{A_1} := \sup_Q \langle w\rangle_Q \|w^{-1}\|_{L^\infty(Q)},$$

$$[w]_{A_p} := \sup_Q \langle w\rangle_Q \langle w^{1-p'}\rangle_Q^{p-1},\ \text{for } p\in(1,\infty)$$

and

$$[w]_{A_\infty} := \sup_Q \langle w\rangle_Q \exp(\langle \log w^{-1}\rangle_Q).$$

Remark 2.2.

(a) $[w]_{A_p}\geq 1$ and $p<q \Rightarrow A_p\subset A_q$.

(b) Because $\lim_{r\searrow 0}\langle |f|^r\rangle_Q^{1/r} = \exp\langle \log|f|\rangle_Q$, it follows $\lim_{p\nearrow\infty}[w]_{A_p} = [w]_{A_\infty}$.

Definition 2.3. With a weight w, one defines $BMO(w)$ as the space of locally integrable functions f with respect to w such that

$$\|f\|_{BMO(w)} = \sup_Q \langle |f-\langle f\rangle_{Q;w}|\rangle_{Q;w} < \infty.$$

Remark 2.4. There is another weighted BMO, BMO_w, which is defined by

$$\|f\|_{BMO_w} = \sup_Q \inf_{c\in\mathbf{C}} \frac{1}{w(Q)}\int_Q |f-c|dx<\infty.$$

It is known that for $w\in A_\infty$, this space is the dual space of the weighted Hardy space $H^1(w)$, i.e., $BMO_w = (H^1(w))^*$, see [3].

The definition of Wilson's constant $\|w\|_{A_\infty}$ uses the restricted Hardy-Littlewood maximal operator.

Definition 2.5. For any measurable subset $E\subset\mathbf{R}^n$, Hardy-Littlewood maximal operator M_E restricted to E is defined by

$$M_E f(x) = \sup_{E\supset R\ni x}\langle |f|\rangle_R,$$

where the supremun is taken over all cubes R containing x and included in E. When $E=\mathbf{R}^n$, we write $M=M_E$.

Definition 2.6.

$$\|w\|_{A_\infty} = \sup_Q \frac{1}{w(Q)}\int_Q M_Q w dx.$$

Remark 2.7.

(a) $w\in A_\infty \Longleftrightarrow \|w\|_{A_\infty}<\infty$, and $\|w\|_{A_\infty}\leq c_n[w]_{A_\infty}$.

(b) There are several equivalent quantities to $\|w\|_{A_\infty}$;

$$\begin{aligned}\|w\|_{A_\infty} &\approx \sup_Q \frac{1}{w(Q)}\int_Q w\log\left(e+\frac{1}{\langle w\rangle_Q}\right)dx\\ &\approx \sup_Q \frac{1}{\langle w\rangle_Q}\|w\|_{L\log L(Q)}\\ &\approx \sup_Q \frac{1}{w(Q)}\int_{2Q} M(\chi_Q w)dx\\ &\approx \sup_Q \frac{1}{w(Q)}\int_{2Q}|R_j(\chi_Q w)|dx,\end{aligned}$$

where $j=1,\cdots,n$, $\|f\|_{L\log L(Q)}$ is defined by

$$\inf\left\{\lambda>0; \left\langle \frac{|f|}{\lambda}\log\left(e+\frac{|f|}{\lambda}\right)\right\rangle_Q \leq 1\right\}$$

and R_j is the j-th Riesz transformation. The first and second equivalences are proved by $L\log L$ theory due to Stein [10]. The third and fourth ones were proved by Fujii [2]. From the third representation, we obtain an inequality

$$M(\chi_Q w)(2Q)\leq c_n\|w\|_{A_\infty} w(Q),$$

which should be compared with the doubling inequality with $[w]_{A_\infty}$;

$$w(2Q)\leq 2^{2^n}[w]_{A_\infty}^{2^n} w(Q),$$

see for example [4].

3. Lower bound. Owing to a version of John-Nirenberg inequality in the context of non-doubling measures in [8], one obtains a variant of the equivalence

$$\|f\|_{BMO}\approx \sup_Q \|f-\langle f\rangle_Q\|_{\exp L(Q)} \tag{3.1}$$

with constants independent of weights.

Lemma 3.1. *There exist constants* $c_1, c_2>0$ *such that for any* $w\in A_\infty$, *it follows*

$$c_1 \sup_Q \|f - \langle f\rangle_{Q;w}\|_{\exp L(Q;w)} \le \|f\|_{BMO(w)} \le c_2 \sup_Q \|f - \langle f\rangle_{Q;w}\|_{\exp L(Q;w)},$$

where $\|f\|_{\exp L(Q;w)}$ *is defined by*

$$\inf\left\{\lambda > 0; \left\langle \exp\left(\frac{|f|}{\lambda}\right) - 1\right\rangle_{Q;w} \le 1\right\}.$$

With this lemma, we give a proof of our lower bound, Theorem 1.1.

Proof of Theorem 1.1. From the definition of $\|f\|_{\exp L(Q;w)}$ above, it follows

$$\left\langle \exp\left(\frac{|f|}{\|f\|_{\exp L(Q;w)}}\right)\right\rangle_{Q;w} \le 2.$$

By using the version of Jensen's inequality

$$\exp\langle g\rangle_Q \le [w]_{A_\infty} \langle \exp(g)\rangle_{Q;w}, \tag{3.2}$$

one obtains

$$\langle |f|\rangle_Q \le \log(2[w]_{A_\infty})\|f\|_{\exp L(Q;w)}.$$

The proof is completed by this inequality and Lemma 3.1 as follows:

$$\begin{aligned}\langle |f - \langle f\rangle_Q|\rangle_Q &\le 2\langle |f - \langle f\rangle_{Q;w}|\rangle_Q \\ &\le 2\log(2[w]_{A_\infty})\|f - \langle f\rangle_{Q;w}\|_{\exp L(Q;w)} \\ &\le c_n \log(2[w]_{A_\infty})\|f\|_{BMO(w)}.\end{aligned}$$

□

Remark 3.2. The inequality (3.2) is equivalent to

$$\exp\langle \log|f|\rangle_Q \le [w]_{A_\infty}\langle |f|\rangle_{Q;w}, \tag{3.3}$$

which should be compared with (4.1). (3.3) can be verified by taking $p \nearrow \infty$ in

$$\langle |f|^{1/p}\rangle_Q^p \le [w]_{A_p}\langle |f|\rangle_{Q;w},$$

see 2 in Remark 2.2.

4. Two other proofs of the upper bound.

Here, we give two other proofs of the upper bound without reverse Hölder inequality.

4.1. Method based on a dual inequality.

The key inequality in this method is the following dual inequality with local sharp maximal operator due to Lerner [7];

Proposition 4.1. *There exists* $c_n > 0$ *so that for any* $\lambda < c_n$

$$\frac{1}{|Q|}\int_Q |f - \langle f\rangle_Q| g\,dx \le c_n \int_Q M_\lambda^\sharp f M_Q g\,dx,$$

where $M_\lambda^\sharp f(x) = \sup_{Q\ni x}\inf_{c\in\mathbf{C}} (\chi_Q(f-c))^*(\lambda|Q|)$, $(0 < \lambda < 1)$ *and* g^* *means the non-increasing rearrangement of* g.

Using this proposition, we can immediately show the optimal upper bound (1.1) as follows:

Proof of (1.1).

$$\begin{aligned}\langle |f - \langle f\rangle_{Q;w}|\rangle_{Q;w} &\le 2\langle |f - \langle f\rangle_Q|\rangle_{Q;w} \\ &\le c_n \frac{1}{w(Q)}\int_Q M_\lambda^\sharp f M_Q w\,dx \\ &\le c_n \|f\|_{BMO}\|w\|_{A_\infty}.\end{aligned}$$

□

4.2. Method based on another representation of $\|w\|_{A_\infty}$.

Next, we give a proof of (1.1) by using another representation of $\|w\|_{A_\infty}$.

Proposition 4.2.

$$\|w\|_{A_\infty} \approx \sup_{Q,f}\frac{\langle |f|\rangle_{Q;w}}{\|f\|_{\exp L(Q)}},$$

where $\|f\|_{\exp L(Q)}$ *is defined by*

$$\inf\left\{\lambda > 0; \left\langle \exp\left(\frac{|f|}{\lambda}\right) - 1\right\rangle_Q \le 1\right\}.$$

Remark 4.3. This form should be compared with

$$[w]_{A_\infty} = \sup_{Q,f}\frac{\exp\langle \log|f|\rangle_Q}{\langle |f|\rangle_{Q;w}},$$

see for example [3].

We show this proposition and then give a proof of (1.1).

Proof. By Hölder inequality in the context of Orlicz spaces, we have

$$\begin{aligned}\langle |f|\rangle_{Q;w} &\le c_n \frac{|Q|}{w(Q)}\|f\|_{\exp L(Q)}\|w\|_{L\log L(Q)} \\ &\le c_n\|w\|_{A_\infty}\|f\|_{\exp L(Q)}.\end{aligned}$$

On the other hand, for a cube Q, from the duality, we can find a function $g \in \exp L(Q)$ such that

$$\begin{aligned}\|w\|_{L\log L(Q)}\|g\|_{\exp L(Q)} &\le c_n \frac{1}{|Q|}\left|\int_Q wg\,dx\right| \\ &\le c_n\langle w\rangle_Q\langle |g|\rangle_{Q;w},\end{aligned}$$

and then, by using the representation of $\|w\|_{A_\infty}$ in Remark 2.7, one obtains

$$\|w\|_{A_\infty} \le c_n \sup_Q \frac{1}{\langle w\rangle_Q}\|w\|_{L\log L(Q)}$$

$$\leq c_n \sup_{Q} \frac{\langle |g| \rangle_{Q;w}}{\|g\|_{\exp L(Q)}}$$
$$\leq c_n \sup_{Q,f} \frac{\langle |f| \rangle_{Q;w}}{\|f\|_{\exp L(Q)}}.$$

□

Proof of (1.1). From Proposition 4.2, it holds

$$\langle |f| \rangle_{Q;w} \leq c_n \|w\|_{A_\infty} \|f\|_{\exp L(Q)}. \tag{4.1}$$

Therefore,

$$\begin{aligned}\langle |f - \langle f \rangle_{Q;w}| \rangle_{Q;w} &\leq 2 \langle |f - \langle f \rangle_Q| \rangle_{Q;w} \\ &\leq c_n \|w\|_{A_\infty} \|f - \langle f \rangle_Q\|_{\exp L(Q)} \\ &\leq c_n \|w\|_{A_\infty} \|f\|_{BMO}.\end{aligned}$$

□

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