On some Hasse principles for algebraic groups over global fields. II

By Ngô Thị NGOAN*) and Nguyễn Quốc THẮNG**)

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Abstract: In this paper, we prove the validity of the cohomological Hasse principle for H^1 of semisimple simply connected algebraic groups defined over infinite algebraic extensions of global fields and also some local–global principles for (skew-)hermitian forms defined over such fields.

Key words: Hasse principle; algebraic groups.

Introduction. In the past, along with the study of local (resp. global) fields and their finite extensions, there is a long and rich history of corresponding study of *infinite* algebraic extensions of local and global fields. In fact, this is indispensable, since there is a very close relation between the studies of finite extensions of local and global fields and that of infinite extensions of those fields. For example, one may cite the study of the maximal unramified extension of a local field or the maximal abelian extension of a global field. The main focus in such study was the divisorial theory (i.e. the arithmetic) of infinite global fields in general, and local and global class field theory, in particular, say as it was done in the works by Albert, Artin and Tate, Herbrand, Kawada, Krull, Moriya, Nakayama and Schilling, Scholz, Stiemke, Whaples etc., which dated back as early as 1926 (Stiemke). The readers can consult [Schi], [AT] and [Ka] and references there as a main record of this period. Later on, there are also some works done, notably on local fields and their extensions (see [Fe] and references there for a record) and on asymptotic and analytic theory of global fields, say by Tsfasman and Vladuts ([TV]) etc. In general, the arithmetic theory of infinite local or global fields is somehow mysterious and it is worth investigating.

One of the famous local–global principles, is Hasse–Minkowski Theorem, which is an important result in number theory. To our knowledge, the validity of an analog of the Hasse–Minkowski Theorem for quadratic forms over infinite algebraic extensions of global fields was for the first time discussed in [KK]. (For pseudoglobal fields it was discussed in [An].) One must note that the relation : *global versus completions* in the classical setting is replaced by another one : *global versus localizations*, where, for a place v of an infinite algebraic extension k of a global field L, the localization field is a certain subfield $k(v)$ contained in the completion k_v of k at v, see below. Further, to distinguish these two approaches, the setting where completions are involved is called *classical Hasse principle*. In particular, it was shown that for quadratic forms in $n \geq 3$ variables, the theorem still holds in new setting, and in general, it may fail for forms of fewer variables. Thus, the classical Hasse–Minkowski Theorem may not hold for quadratic forms in the case of infinite extensions of global fields.

This paper is the sequel to our paper [NT1], where we consider some Hasse principles for algebraic groups over global fields. Our goal in this note is to see to what extent one can extend the known results for (cohomological) Hasse principles over ordinary global fields to their *infinite algebraic extensions*. One should note that the arithmetic results obtained can be varying quite differently depending on what kind of fields we are dealing with. For example, if k is just an algebraic closure of a global field, then the arithmetic results are almost trivial in this case. Our main results are the validity of the cohomological Hasse principle for H^1 of semisimple simply connected algebraic groups defined over an infinite algebraic extension of a global field and also some local–global principles for (skew-)hermitian forms over such fields. In another paper under preparation, we consider some arithmetic questions of algebraic groups related with infinite local and global fields. The details will be published elsewhere (see [NT2]).

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*) Department of Mathematics and Informatics, College of Science, Thai Nguyen University, Thainguyen, Vietnam.
**) Institute of Mathematics, VAST, 18-Hoang Quoc Viet, Hanoi, Vietnam.

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Conventions and notation. An infinite algebraic extension of a local (resp. global) field will be called shortly just infinite local (resp. global) field. If k is a field, V_k denotes the set of all places of k. If G is an affine algebraic group defined over a field k, then $\mathrm{H}^i_{flat}(k, G)$ denotes the i-th flat cohomology of G (where $i \leq 1$, if G is non-commutative). If $S \subset V_k$, we say that a k-group G has weak approximation property with respect to S, if, via the diagonal embedding, $G(k)$ is dense in $\prod_{v \in S} G(k_v)$ in the product topology. We say that the cohomological Hasse principle (in classical setting) holds in dimension i for a k-group G if the natural map $\mathrm{H}^i_{flat}(k, G) \to \prod_{v \in V_k} \mathrm{H}^i_{flat}(k_v, G)$ has trivial kernel, whenever it makes sense.

1. Local–global principle for quadratic forms.

1.1. Localization fields. Let k be an infinite global field, which is an infinite algebraic extension of a global field F. Given such k, F, let v be a valuation of k and let k_v be a completion of k at v. The restrictions of v to finite extensions $F \subset L \subset k$ give rise to the completions of such extensions in a completion k_v, the collection of all these is denoted by $\mathcal{C}$. We say after [KK], that a field k', is a *localization of k at v*, if k' is the direct limit of all extensions from $\mathcal{C}$ and we denote it by $k(v)$. We call such fields as localization fields. Such fields were considered before for the first time by Moriya (see e.g. [Mo1], [Mo2]), Moriya and Schilling, Moriya and Nakayama (see [Schi, Chap. VI, Sec. 11] and reference there).

Just like above, if $S \subset V_k$ is a subset of places of an infinite global field k, we say that a k-group G has weak approximation property (in new setting) with respect to S, if, via the diagonal embedding, $G(k)$ is dense in $\prod_{v \in S} G(k(v))$ in the product topology, where $G(k(v))$ is endowed with the v-adic topology. We say that the cohomological Hasse principle (in new setting) holds in dimension i for a k-group G if the natural map $\mathrm{H}^i_{flat}(k, G) \to \prod_{v \in V_k} \mathrm{H}^i_{flat}(k(v), G)$ has trivial kernel, whenever it makes sense.

1.2. Local classification of quadratic forms over (infinite) local or quasi-finite fields. Let k be an infinite algebraic extension of a finite (resp. local) field F of char. $\neq 2$, f a regular quadratic form over k with dimension n i.e., the number of variables is n. From Springer Theorem, we derive the following statement, the proof of which is essentially the same as in the finite (resp. local) field case.

1.2.1. Theorem. *a) If F is algebraic over a finite field of char. $\neq 2$ and if $n \geq 3$ then any quadratic form f in n variables is isotropic over F. Thus any two quadratic forms over F are equivalent over F if they have the same dimension and discriminant.*
b) If k is a localization with respect to a non-archimedean valuation of an infinite global field, with κ the residue field of char. $\neq 2$. and if $n \geq 5$, then f is isotropic over k.

1.2.2. Lemma. *Let D be a non-trivial quaternion division algebra over a non-archimedean localization field k. Then D is the unique (up to an isomorphism) quaternion division algebra over k.*

1.3. Global classification of quadratic forms. Strong and weak Hasse principle. Let k be an infinite global field, $\mathcal{C}$ the set of all localizations of k. We say that the Hasse principle holds for certain property P, which can be defined over k and over all of its localizations $\in \mathcal{C}$, if P holds over k, whenever P holds over all of the localizations. (Then we say also that P holds *locally everywhere.*) Further, whenever it says that the Hasse principle holds, it is assumed to be in the new setting (i.e. in the sense "global versus localizations"). Here we are interested in the Hasse principle for a non-degenerate quadratic form f over k. We have the following Strong and Weak Hasse principles.

1.3.1. Theorem (Strong Hasse principle [KK]). *Let k be an infinite global field of char. $\neq 2$.*
1) If f is a quadratic form in $n \geq 3$ variables, then the Hasse principle holds for f, i.e., if f represents 0 locally everywhere, then it does so over k.
2) If f is a quadratic form in $n \geq 2$ variables, which represents an element $a \in k$ locally everywhere, then so does it over k.

Remark. The Strong Hasse principle (in new setting) for quadratic forms breaks down exactly when $n = 2$, namely it is possible that the local–global principle does not hold for the quadratic form $x^2 - dy^2$ ([KK]). Thus the cohomological Hasse principle (in new setting) may not hold for $\mathrm{H}^1(k, \mu_2)$, $\mathrm{H}^1(k, \mathrm{O}(f))$, so there are quadratic forms (resp. finite group schemes) which satisfy the classical Hasse–Minkowski (resp. the cohomological Hasse principle) over any (finite) global fields, but the classical Hasse–Minkowski (resp. cohomological Hasse) principles may fail for them over infinite global fields.

2. Hasse Norm Theorem and Hasse–Brauer–Noether Theorem. Let k be an infinite global field, V_k the set of all places of k. We discuss two of the most important local–global principles in the classical arithmetic of the global fields, namely the Hasse Norm Theorem and Hasse–Brauer–Noether Theorem.

2.1. We have the following extension of the (quadratic) Hasse Norm Theorem to our case.

2.1.1. Theorem (Hasse Norm Theorem)**.** *Let k be an infinite global field of char. $\neq 2$.*
1) *Let $K = k(\sqrt{a})$ be a quadratic extension of k. Then an element $b \in k$ is a norm from K if and only if it is so locally everywhere.*
2) *Let D be a quaternion algebra over k. Then D is trivial if and only if it is trivial locally everywhere.*

2.2. Hasse–Brauer–Noether Theorem. One of the celebrated results of global class field theory is Hasse–Brauer–Noether Theorem. We have a weaker form in the infinite global field case.

2.2.1. Theorem (Hasse–Brauer–Noether Theorem)**.** *Let k be an infinite global field. Then the canonical homomorphism $Br(k) \to \prod_v Br(k(v))$ is injective.*

Now we consider central simple algebras with involutions over infinite local or global fields. We have the following.

2.2.2. Proposition. *Let k be an infinite local or global field with $char.k \neq 2$. If A is a central simple k-algebra with a non-trivial involution, then either $A \simeq M_n(D)$, where D is a quaternion division algebra, or A is trivial on k.*

Remark. 1) We can show that not every central simple algebra over an infinite global field is almost trivial locally everywhere (over $k(v)$) (i.e., trivial outside a finite set of places).
2) It remains an open question if the natural map $Br(k) \to \prod_v Br(k_v)$ is injective when $char.k = 0$, while it is true if $char.k > 0$ (see [Mi, Chap. I, Appendix]).

3. Local theory of hermitian forms.

3.1. We classify the quadratic or hermitian forms according to the Dynkin types A, B, C, or D of their corresponding special unitary groups. From the general results of [Sch1], we derive the following local classification of hermitian forms.

3.2. Local theory for forms of Type A. We start with the following classification theorem for forms of type A over local fields. In the case of infinite extensions of local fields we have a result similar to Kneser Theorem over local fields, which will be needed in the sequel.

3.2.1. Theorem. *Let k be a non-archimedean localization field of char. $\neq 2$, D a division k-algebra with center $K = k(\sqrt{a})$ and involution J of second kind, non-trivial on K and h a non-degenerate hermitian form of rank n with respect to J and with values in D. Then $D = K$ and thus h is Morita equivalent to a quadratic form of dimension $2n$ over k.*

3.2.2. Corollary. *With notation as in* 3.2.1*, every hermitian form h of dimension $n \geq 3$ is isotropic over k.*

3.3. Local theory for forms of Type C. We have the following

3.3.1. Proposition. *Let k be a non-archimedean localization field of char. $\neq 2$, h a non-degenerate hermitian form with values in a quaternion division algebra D with standard involution J over k with residue field of char. $\neq 2$. If $dim(h) \geq 2$, then h is isotropic.*

3.4. Local theory for forms of Type D. We have the following analogies of Tsukamoto's result.

3.4.1. Theorem ([Sch2, p. 363] for $\mathfrak{p}$-adic fields)**.** *Let k be a non-archimedean localization field of char. $\neq 2$, h a non-degenerate skew-hermitian form with values in a quaternion division algebra D with standard involution J over k with residue field of char. $\neq 2$. If $dim(h) \geq 4$, then h is isotropic.*

3.4.2. Theorem (cf. Tsukamoto [Tsu], [Sch2, p. 363] for $\mathfrak{p}$-adic fields)**.** *Let k be a non-archimedean localization field of char. $\neq 2$, h a non-degenerate skew-hermitian form with values in a quaternion division algebra D with standard involution J over k with residue field of char. $\neq 2$.*

1) *For any $d \in k^*, d \not\equiv -1(k^{*2})$, there is a skew-hermitian form $h = \langle \alpha \rangle$ with determinant d. If $n = dim(h) > 1$, there is a form of dimension n with determinant any given $d \in k^*$.*

2) *If $n = 1$, the isometric class of h is defined by the determinant $det(h)$. If $n = 2$, h is isotropic if and only if $det(h) = 1(mod.k^{*2})$. If $n = 3$, h is anisotropic if and only if $det(h) = 1(mod.k^{*2})$.*

3) *Non-degenerate skew-hermitian forms are isometric if and only if they have the same dimension and determinant.*

4. Hasse principle and global classification of hermitian forms.

4.1. Strong and weak Hasse principles. In the infinite global field case, we have a complete analog to the well-known Hasse principle over global fields for forms of type A, proved by W. Landherr in 1938 ([Sch2, p. 373], cf. also [Kn, Chap. 5]), for forms of type C (an easy consequence of the Hasse–Minkowski principle for quadratic forms) and for forms of type D (proved by Kneser ([Kn, p. 128], [Sch2, p. 366])).

4.1.1. Theorem (Strong and Weak Hasse principle)**.** *Let k be an infinite global field of char. $\neq 2$, V_k the set of all its places, ∞_k the set of all archimedean places and let g, h be a non-degenerate quadratic or* (*skew-*)*hermitian forms of dimension n and of the same type over a division algebra D* (*of center $K = k(\sqrt{a})$, $k = K^J$, if the form are of type* A).

1) *Assume that $n \geq 3$ if h is of type* D. *Then the form h represents zero over k if and only if it does so locally everywhere.*

2) *Assume that the forms g and h are neither of type* D, *nor quadratic form. Then they are equivalent over k if and only if they are so locally everywhere.*

3) *Assume that the forms g and h are of the same determinant. Then they are equivalent over k if and only if they are so locally everywhere.*

4) (Classical Hasse principle) *Assume that $n \geq 5$. Then the form h represents zero over k if and only if it does so locally everywhere over $k_v, v \in \infty_k$.*

Remark. It is well-known that in the case $n = 2$, Strong (and Weak) Hasse principle may fail for skew-hermitian forms over global fields. It appears that the same failure also happens in the case of infinite global fields, see below.

4.2. Failure of the Strong and Weak Hasse principle for skew-hermitian forms of type D. In contrast to the Strong Hasse principle in case dimension ≥ 3, as in the classical case, the Strong Hasse principle does not hold for two dimensional forms of type D and neither does Weak Hasse principle hold for forms of type D in general. We have

4.2.1. Proposition (Failure of Strong and Weak Hasse principles)**.** 1) *There exist an infinite global field k of char. $\neq 2$, a quaternion division algebra D with the standard involution J over k such that D is defined over a global field L, and $D \otimes K_v$ is not trivial for at least four places v of K for any finite subextension $L \subset K \subset k$.*

2) *For such a field k and quaternion division algebra D, there are skew-hermitian forms h* (*resp. skew-hermitian forms g and h*) *with values in D with respect to J, which are isotropic locally everywhere, but not globally isotropic* (*resp. isometric locally everywhere, but not globally isometric*) *over k.*

3) *There are infinite global fields, such that for any natural number n, there are quadratic forms f and g of dimension n, which are equivalent locally everywhere, but are not so over k.*

5. Hasse principle for Galois cohomology of simply connected groups.

5.1. Analog of Kneser–Bruhat–Tits theorem. We have the following analog of an important result of Galois cohomology of algebraic groups over local fields.

5.1.1. Theorem. *Let $k(v)$* (*resp. k_v*) *be the non-archimedean localization field* (*resp. completion*) *of an infinite algebraic extension of a global field k and G a semisimple simply connected algebraic group defined over $k(v)$. Then $\mathrm{H}^1(k(v), G) = \mathrm{H}^1(k_v, G) = 0$.*

Remark. It was noticed, that many results obtained in the theory of Bruhat–Tits remain valid also for henselian discretely valued field. It is natural to ask if $\mathrm{H}^1(k, G) = 0$ for any simply connected semisimple k-group G, where k is just henselian and has its residue field of cohomological dimension ≤ 1 (cf. [BrT]).

5.2. Infinite global field case. The cohomological Hasse principle for simply connected groups over global fields was established by Kneser, Harder and Chernousov. Now we are able to extend the classical Hasse principle for semisimple simply connected groups over global fields to infinite global fields. First we need the following

5.2.1. Theorem. *Let G be a connected affine algebraic group defined over an infinite global field k and ∞_k the set of archimedean places of k.*

1) *G has weak approximation with respect to the set of all real places, i.e., $G(k)$ is dense in the product $\prod_{v \in \infty_k} G(k_v)$.*

2) *The localization map*

$$loc_S : \mathrm{H}^1(k, G) \to \prod_{v \in S} \mathrm{H}^1(k(v), G) = \prod_{v \in S} \mathrm{H}^1(k_v, G)$$

is surjective for any finite set $S \subset \infty_k$.

The cohomological Hasse principle for semisimple simply connected algebraic groups over global fields was proved by Kneser, Harder and Chernousov (cf. [PlR, Chap. 6], [Ha2], [Kn]). We have the following extension to the case of infinite global fields, which is one of the main results of the paper.

5.2.2. Theorem (Hasse principle for simply connected groups). *Let k be an infinite algebraic extension of a global field L, $k = \cup_n L_n, [L_n : L] < \infty$, and G a connected smooth affine algebraic group scheme defined over L.*
1) *If the (classical) cohomological Hasse principle holds for G over every finite extension $L \subset L_n \subset k$, then the same holds for G over k.*
2) *If G is a semisimple simply connected algebraic group defined over k then the cohomological Hasse principle holds for G.*
3) *If G is as in 2), then we have natural injection*

$$\gamma : \mathrm{H}^1(k, G) \hookrightarrow \prod_{v \in \infty_k} \mathrm{H}^1(k(v), G) \simeq \prod_{v \in \infty_k} \mathrm{H}^1(k_v, G).$$

In particular, the classical cohomological Hasse principle also holds for G. The set $\infty'_k := \{v \in \infty_k \mid \mathrm{H}^1(k_v, G) \neq 0\}$ is finite if an only if $\mathrm{H}^1(k, G)$ is finite. If it is so, then γ is bijective.

Recall that if k is an imaginary global field (in particular, if *char.*$k > 0$), then by a theorem of Harder [Ha1, Haupsatz], [Ha2, Satz A]), for any semisimple simply connected k-group G, we have $\mathrm{H}^1(k, G) = 0$. The same also holds in the infinite global field case, as the following shows.

5.2.3. Corollary. *Let k be an infinite global field and G a semisimple simply connected group defined over k. If k is imaginary (in particular, if char.$k > 0$), then $\mathrm{H}^1(k, G) = 0$.*

Remark. 1) In the situation of Theorem 5.2.2, 1), one can say also that if a connected smooth affine algebraic group defined over a global subfield L of an infinite global field k satisfies the cohomological Hasse principle "locally" everywhere, that is, for some subfield K, over each global subfield $K \subset L$ contained in k, then it does so over k.
2) As it was mentioned above, the classical cohomological Hasse principle for H^1 may fail for certain non-connected algebraic groups over infinite global fields, though such groups do satisfy the Hasse principle over any global fields. So the connectedness plays an essential role here.

6. Counter-examples to the Hasse principle for connected groups. It is natural to investigate the analogous phenomenon for connected algebraic groups over infinite global fields. Let $\mathcal{F}$ be the class of fields consisting of (possibly infinite) global fields k such that for any given natural number n, there exist finite odd degree extensions K/k with $([K : k], n) = 1$. We say that k is of type $\mathcal{F}$ if $k \in \mathcal{F}$. From the global class field theory, we know that any global field belongs to $\mathcal{F}$.

6.1. Injectivity of the restriction maps. We need some results regarding restriction maps for Galois cohomology. Let k be a field and let G be a reductive k-group. The following result extends a result by Sansuc in the number field case to function field case.

6.1.1. Proposition (Sansuc [Sa, Corol. 4.6] for number fields). *Let k be a (possibly infinite) non-algebraically closed either non-archimedean local or global field of type $\mathcal{F}$ with ∞_k finite and G a connected reductive k-group. Then for any given positive number N, there exists a finite Galois extension K/k of degree $> N$ such that the restriction map $\mathrm{H}^1(k, G) \to \mathrm{H}^1(K, G)$ is injective.*

As an immediate consequence, we have the following

6.1.2. Corollary ([Sa, Corol. 4.7, 4.8] for number fields). *Let k be a (possibly infinite) non-algebraically closed either non-archimedean local or global field of type $\mathcal{F}$ with ∞_k finite and let G a connected reductive k-group. Then*
1) *There exists a natural number $m(G) > 1$, such that if K/k is a finite extension of degree n, where $(n, m(G)) = 1$, then the restriction map $\mathrm{H}^1(k, G) \to \mathrm{H}^1(K, G)$ is injective.*
2) *Let $n_1, \dots, n_s > 1$ be coprime natural numbers and let k_i/k be a finite extension of degree n_i. Then the natural map $\mathrm{H}^1(k, G) \to \prod_i \mathrm{H}^1(k_i, G)$ is injective.*
In particular, every G-torsor which is defined over k having a rational point (resp. points) in K (in case 1)) (resp. in each of k_i, in case 2)), has already a k-point.

6.2. Construction of contre-examples. By Theorem 5.2.2, if G is a connected affine algebraic group defined over an infinite global field k, for which the cohomological Hasse principle does not hold, then there exists some global subfield $L \subset k$, such that G is defined over L and does not satisfy the (classical) cohomological Hasse principle over L. There are examples of connected groups over global fields for which the cohomological Hasse principle fails, cf. e.g. [Sa], [Se] etc. Here we prove the converse statement.

6.2.1. Proposition. *Let L be a global field and G a connected reductive L-group, such that for G the cohomological Hasse principle (for H^1) fails for G over L. Then there exists an infinite algebraic extension k/L such that $G \times_L k$ does not satisfy the cohomological Hasse principle for H^1 over k.*

7. The proofs. In the local case, the proof makes use of the classical theory of forms over local fields together with Scharlau results [Sch1,Sch2]. In the global case, one makes use of the classical theory over global fields together with a lemma due to König ([Kö, Thm. E]), following a method of [KK].

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