Euler products beyond the boundary for Selberg zeta functions

By Shin-ya KOYAMA[*)] and Fumika SUZUKI[**)]

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Abstract: Convergence of Euler products in the critical strip is directly related to a proof of the generalized Riemann hypothesis. Moreover its behavior on the critical line is called the deep Riemann hypothesis (DRH). Kimura-Koyama-Kurokawa recently proved DRH over function fields in case the L-function is regular at $s = 1$ [3]. In this paper we generalize their results to Selberg zeta functions. Our results imply the DRH for principal congruence groups over function fields.

Key words: Selberg zeta functions; Riemann hypothesis; Euler products.

1. Introduction. The Euler product for the Riemann zeta function

$$\zeta(s) = \prod_{p:\,\mathrm{prime}} (1 - p^{-s})^{-1}$$

is absolutely convergent in $\mathrm{Re}(s) > 1$ and divergent in $\mathrm{Re}(s) < 1$. For proving this divergence we need its pole at $s = 1$. Indeed, if we eliminate the pole by considering a Dirichlet L-function

$$L(s,\chi) = \prod_{p:\,\mathrm{prime}} (1 - \chi(p)p^{-s})^{-1},$$

where χ is a Dirichlet character with $\chi \neq \mathbf{1}$ with $\mathbf{1}$ denoting the principal character, its convergence in $0 < \mathrm{Re}(s) < 1$ is an unsolved problem. In this region we have to take care of the order of the Euler factors participating in the Euler product, since it is not absolutely convergent. So putting

$$L_t(s,\chi) = \prod_{p<t:\,\mathrm{prime}} (1 - \chi(p)p^{-s})^{-1},$$

we consider its convergence as $t \to \infty$.

Conjecture 1. For any χ and $s \in \mathbf{C}$ such that $\chi \neq \mathbf{1}$ and $\mathrm{Re}(s) > \frac{1}{2}$, it holds that

$$\lim_{t\to\infty} L_t(s,\chi) = L(s,\chi).$$

Whenever an infinite product converges, it is non-zero by definition. Therefore Conjecture 1 implies the (generalized) Riemann hypothesis for $L(s,\chi)$. It is known by Conrad [3] that Conjecture 1 is implied by the following conjecture.

Conjecture 2. For any χ and $s \in \mathbf{C}$ such that $\chi \neq \mathbf{1}$, $\mathrm{Re}(s) = \frac{1}{2}$ and $L(s,\chi) \neq 0$, it holds that

$$\lim_{t\to\infty} L_t(s,\chi) = \begin{cases} \sqrt{2}L(s,\chi) & (\chi^2 = 1,\ s = \frac{1}{2}) \\ L(s,\chi) & (\text{otherwise}). \end{cases}$$

Conjecture 2 is called the deep Riemann hypothesis (DRH), because it is stroger than Conjecture 1 (RH).

In the previous paper [4], we proved an analog of Conjecture 2 over function fields. Let $\mathbf{F}_q$ be the finite field of q elements. We fix a conductor $f(T) \in \mathbf{F}_q[T]$ and introduce a "Dirichlet" character

$$\chi:\ (\mathbf{F}_q[T]/(f))^\times \to \mathbf{C}^\times.$$

We define the "Dirichlet" L-function as

$$L_{\mathbf{F}_q(T)}(s,\chi) = \prod_h (1 - \chi(h)N(h)^{-s})^{-1},$$

where $h = h(T) \in \mathbf{F}_q[T]$ runs through monic irreducible polynomials, and $N(h) = q^{\deg h}$. We proved the following theorem in [4], which was extended to automorphic L-functions in [7].

Theorem 3 (DRH over function fields)**.** *Let q, f and χ be as above. Put $K = \mathbf{F}_q(T)$ and assume $\chi \neq \mathbf{1}$. Then the following* (1) *and* (2) *are true.*

(1) *For* $\mathrm{Re}(s) > 1/2$, *we have*

$$\lim_{n\to\infty} \prod_{\deg h \leq n} (1 - \chi(h)N(h)^{-s})^{-1} = L_K(s,\chi).$$

(2) *For $t \in \mathbf{R}$ with $L_K(\frac{1}{2} + it, \chi) \neq 0$, it holds that*

$$\lim_{n\to\infty} \prod_{\deg h \leq n} (1 - \chi(h)N(h)^{-\frac{1}{2}-it})^{-1} = L_K\left(\frac{1}{2} + it, \chi\right) \times \begin{cases} \sqrt{2} & (\chi^2 = \mathbf{1},\ t \in \frac{\pi}{\log q}\mathbf{Z}) \\ \mathbf{1} & (\textit{otherwise}) \end{cases}.$$

In this paper we generalize this result to Selberg

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*) Department of Biomedical Engineering, Toyo University, 2100 Kujirai, Kawagoe, Saitama 350-8585, Japan.
**) Department of Physics and Astronomy, The University of British Columbia, Vancouver, BC V6T 1Z1, Canada.

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zeta functions.

2. Results for $1/2 < \mathrm{Re}(s) < 1$. We start from a general Euler product:

$$(1)\qquad L_X(s,\rho) = \prod_{p\in \mathrm{Prim}(X)} \det(I_n - \rho(p)N(p)^{-s})^{-1},$$

where $\mathrm{Prim}(X)$ is a set of prime elements in some mathematical object X, ρ is a map from $\mathrm{Prim}(X)$ to $U(n)$ with n being a positive integer, and N is a map from $\mathrm{Prim}(X)$ to $\mathbf{R}_{>1}$. We assume that (1) is absolutely convergent in $\mathrm{Re}(s) > 1$. The Riemann zeta or Dirichlet L-functions ($X = \mathbf{Z}$, $\mathrm{Prim}(\mathbf{Z}) = \{\text{prime numbers}\}$, $N(p) = p$, $\rho =$ Dirichlet character, $n = 1$) and the Selberg zeta function ($X = \Gamma$: a Fuchsian group, $\mathrm{Prim}(\Gamma) = \{\text{prime geodesics in } \Gamma\backslash H\}$ with H the upper half plane, $N(p) = e^{\mathrm{lentgh(p)}}$, ρ a unitary representation of Γ, $n = \dim\rho$) satisfy this assumption.

In case X is defined over function fields over $\mathbf{F}_q$, it often happens that $L_X(s,\chi)$ satisfies the following.

Assumption 1. For any $p \in \mathrm{Prim}(X)$, there exists a positive integer d such that $N(p) = q^d$. Moreover $L_X(s,\rho)$ is a rational power of a rational function in q^{-s} which can be put as

$$L_X(s,\rho) = \prod_{j=1}^{J} (1 - \alpha_j q^{-s})^{-v_j}$$

for some $J \in \mathbf{Z}$, $\alpha_j \in \mathbf{C}$ with $|\alpha_j| < q$ and $v_j \in \mathbf{Q}$. The following lemma will be a key to our proof.

Lemma 4. *Let $\lambda_{p,i} \in \mathbf{C}$ with $|\lambda_{p,i}| = 1$ ($i = 1, 2, \cdots, n$) be the eigenvalues of $\rho(p)$. If $L_X(s,\rho)$ satisfies Assumption 1, it holds that*

$$\sum_{j=1}^{J} v_j \alpha_j^m = \sum_{i=1}^{n} \sum_{d|m} d \sum_{\substack{p\\ N(p)=q^d}} \lambda_{p,i}^{\frac{m}{d}} \quad (m = 1, 2, 3, \cdots).$$

Proof. We compute that for $\mathrm{Re}(s) > 1$,

$$\log L_X(s,\rho) = \sum_{j=1}^{J} v_j \log(1 - \alpha_j q^{-s})^{-1} = \sum_{m=1}^{\infty} \frac{q^{-ms}}{m} \sum_{j=1}^{J} v_j \alpha_j^m,$$

where in the last identity we used the assumptions $|\alpha_j| < q$ and $\mathrm{Re}(s) > 1$.

On the other hand, we calculate from (1) that

$$\begin{aligned}\log L_X(s,\rho) &= \sum_{p} \sum_{i=1}^{n} \log(1 - \lambda_{p,i} N(p)^{-s})^{-1} \\ &= \sum_{p} \sum_{i=1}^{n} \sum_{k=1}^{\infty} \frac{\lambda_{p,i}^k}{k} N(p)^{-ks} \\ &= \sum_{d=1}^{\infty} \sum_{i=1}^{n} \sum_{k=1}^{\infty} \frac{q^{-dks}}{k} \sum_{\substack{p\\ N(p)=q^d}} \lambda_{p,i}^k \\ &= \sum_{m=1}^{\infty} \frac{q^{-ms}}{m} \sum_{i=1}^{n} \sum_{d|m} d \sum_{\substack{p\\ N(p)=q^d}} \lambda_{p,i}^k.\end{aligned}$$

Comparing these two expressions leads to the result. □

In what follows let ρ be a unitary representation of $X = \Gamma = \mathrm{PGL}(2, \mathbf{F}_q[T])$. We define the *Selberg zeta function* of Γ attached by a unitary representation ρ as the Euler product (1), where $\mathrm{Prim}(\Gamma)$ is the set of primitive hyperbolic conjugacy classes, and $N(p) = q^{2d}$ with d being the degree of the larger eigenvalue of p. Here p is *hyperbolic* if and only if p has two distinct eigenvalues in $\mathbf{F}_q((T^{-1}))$. It is known that for such Selberg zeta functions, the Euler product (1) is absolutely convergent in $\mathrm{Re}(s) > 1$ [8]. Put

$$L_{\Gamma,x}(s,\rho) = \prod_{\substack{p\\ N(p)\le x}} \det(I_n - \rho(p)N(p)^{-s})^{-1}.$$

Theorem 5. *Assume $1/2 < \mathrm{Re}(s) < 1$. Under Assumption 1, it holds that*

$$\lim_{x\to\infty} L_{\Gamma,x}(s,\rho) = L_\Gamma(s,\rho) \quad \left(\mathrm{Re}(s) > \max_j \log_q |\alpha_j|\right).$$

Proof. We divide the series

$$\log L_{\Gamma,x}(s,\rho) = \sum_{\substack{p\\ N(p)\le x}} \sum_{i=1}^{n} \sum_{k=1}^{\infty} \frac{\lambda_{p,i}^k}{k} N(p)^{-ks}$$

into

$$A(x) = \sum_{k=1}^{\infty} \sum_{\substack{p\\ N(p)\le x^{\frac{1}{k}}}} \frac{N(p)^{-ks}}{k} \sum_{i=1}^{n} \lambda_{p,i}^k$$

and

$$B(x) = \sum_{k=1}^{\infty} \sum_{\substack{p\\ x^{\frac{1}{k}}<N(p)\le x}} \frac{N(p)^{-ks}}{k} \sum_{i=1}^{n} \lambda_{p,i}^k.$$

By Lemma 4, we compute $A(x)$ as

$$A(x) = \sum_{k=1}^{\infty} \sum_{\substack{d\\ q^d\le x^{\frac{1}{k}}}} \frac{q^{-dks}}{k} \sum_{\substack{p\\ N(p)=q^d}} \sum_{i=1}^{n} \lambda_{p,i}^k$$

$$= \sum_{1\le m\le \log_q x} \frac{q^{-ms}}{m} \sum_{d|m} d \sum_{\substack{p \\ N(p)=q^d}} \sum_{i=1}^{n} \lambda_{p,i}^k$$

$$= \sum_{1\le m\le \log_q x} \frac{q^{-ms}}{m} \sum_{j=1}^{J} v_j \alpha_j^m$$

$$\underset{x\to\infty}{\longrightarrow} \sum_{j=1}^{J} v_j \log(1-\alpha_j q^{-s})^{-1} \quad (|\alpha_j q^{-s}|<1)$$

$$= \log \prod_{j=1}^{J} (1-\alpha_j q^{-s})^{-v_j}$$

$$= \log L_\Gamma(s,\rho),$$

where we used the assumption $\sigma > \max_j \log_q |\alpha_j|$ in letting $x\to\infty$. It suffices to show that $B(x)\to 0$ as $x\to\infty$. Since $L_\Gamma(s,\rho)$ is absolutely convergent in $\mathrm{Re}(s)>1$, we have

$$\sum_{p\in \mathrm{Prim}(X)} \sum_{i=1}^{n} |\lambda_{p,i} N(p)^{-s}| = n\sum_p N(p)^{-\sigma} < \infty$$

for $\sigma=\mathrm{Re}(s)>1$. It implies that

$$\lim_{t\to\infty} \sum_{\substack{p \\ N(p)>t}} N(p)^{-\sigma} = 0 \tag{2}$$

for $\sigma>1$. We evaluate $B(x)$ as follows:

$$|B(x)| \le n\sum_{k=2}^{\infty} \sum_{\substack{p \\ x^{\frac{1}{k}}<N(p)\le x}} \frac{N(p)^{-k\sigma}}{k} \qquad (\sigma := \mathrm{Re}(s))$$

$$= n(B_1(x)+B_2(x)+B_3(x))$$

with

$$B_1(x) = \sum_{k=3}^{\infty} \sum_{\substack{p \\ x^{\frac{1}{k}}<N(p)\le x^{\frac{1}{2}}}} \frac{N(p)^{-k\sigma}}{k},$$

$$B_2(x) = \sum_{k=3}^{\infty} \sum_{\substack{p \\ x^{\frac{1}{2}}<N(p)\le x}} \frac{N(p)^{-k\sigma}}{k},$$

$$B_3(x) = \sum_{\substack{p \\ x^{\frac{1}{2}}<N(p)\le x}} \frac{N(p)^{-2\sigma}}{2}.$$

First, we deal with $B_1(x)$ by changing the order of the double sum:

$$B_1(x) = \sum_{\substack{p \\ N(p)\le x^{\frac{1}{2}}}} \sum_{k>\log_{N(p)} x} \frac{N(p)^{-k\sigma}}{k}$$

$$\le \sum_{\substack{p \\ N(p)\le x^{\frac{1}{2}}}} \frac{1}{\log_{N(p)} x} \sum_{k>\log_{N(p)} x} N(p)^{-k\sigma}$$

$$\le \sum_{\substack{p \\ N(p)\le x^{\frac{1}{2}}}} \frac{1}{\log_{N(p)} x} \frac{x^{-\sigma}}{1-N(p)^{-\sigma}}$$

$$\le \sum_{\substack{p \\ N(p)\le x^{\frac{1}{2}}}} \frac{1}{2} \frac{x^{-\sigma}}{1-q^{-\sigma}}$$

$$\le \frac{x^{-\sigma}}{2(1-q^{-\sigma})} \sum_{\substack{p \\ N(p)\le x^{\frac{1}{2}}}} 1.$$

By the prime geodesic theorem, the number of $p\in \mathrm{Prim}(\Gamma)$ with $N(p)<x$ is approximately known:

$$\pi_\Gamma(x) := \#\{p\in \mathrm{Prim}(\Gamma) \mid N(p)\le x\} \sim \frac{x}{\log x} \quad (x\to\infty).$$

Thus

$$B_1(x) = \frac{x^{-\sigma}}{2(1-q^{-\sigma})} \pi_\Gamma(\sqrt{x}) \sim \frac{x^{-\sigma}}{2(1-q^{-\sigma})} \frac{\sqrt{x}}{\log\sqrt{x}} = O\left(\frac{x^{\frac{1}{2}-\sigma}}{\log x}\right),$$

which tends to 0 as $x\to\infty$, if $\sigma>\frac{1}{2}$.

Next we treat $B_2(x)$. It holds that

$$B_2(x) \le \frac{1}{3}\sum_{k=3}^{\infty} \sum_{\substack{p \\ \sqrt{x}<N(p)\le x}} N(p)^{-k/2}$$

$$= \frac{1}{3} \sum_{\substack{p \\ \sqrt{x}<N(p)\le x}} \frac{N(p)^{-3/2}}{1-N(p)^{-1/2}}$$

$$= \frac{1}{3} \sum_{\substack{p \\ \sqrt{x}<N(p)\le x}} \frac{1}{N(p)^{3/2}(1-N(p)^{-1/2})}$$

$$\le \frac{1}{3} \sum_{\substack{p \\ \sqrt{x}<N(p)\le x}} \frac{1}{N(p)^{3/2}(1-q^{-1/2})}$$

$$\le \frac{1}{3(1-q^{-1/2})} \sum_{\substack{p \\ \sqrt{x}<N(p)}} \frac{1}{N(p)^{3/2}}.$$

It follows from (2) that $B_2(x)\to 0$ as $x\to\infty$. Finally, $B_3(x)$ is similarly bounded by the above absolutely convergent function with $s=2\sigma>1$. □

Theorem 5 generally extends the region of Euler products to a subdomain in the critical strip. It agrees to the full strip $1/2<\mathrm{Re}(s)<1$ under the following condition.

Assumption 2. $|\alpha_j| < \sqrt{q}\,(j = 1, 2, 3, \ldots, J)$.

3. Results for $\mathbf{Re}(s) = 1/2$. We need a lemma of Mertens' type as follows:

Lemma 6 (Mertens' theorem)**.** *Assume* $\mathrm{Re}(s) = 1$ *and let ρ be any complex representation of Γ such that $L_\Gamma(s, \rho)$ is rational in q^{-s}. Then*

$$\sum_{\substack{p \\ N(p) \le q^D}} \frac{\mathrm{tr}(\rho(p))}{N(p)^s} = \begin{cases} \mathrm{mult}(\mathbf{1}, \rho) \log D + c_1(\rho) + o(1) & (s \in 1 + \frac{\pi i}{\log q}\mathbf{Z}) \\ c_2(\rho) + o(1) & (\textit{otherwise}) \end{cases}$$

as $D \to \infty$, where $c_1(\rho)$, $c_2(\rho)$ are constants independent of D.

Proof. We follow the proof of Theorem 2.1 in [6]. Write

$$\rho = \mathbf{1}^{\oplus \mathrm{mult}(\mathbf{1}, \rho)} \oplus \rho_0,$$

where $\mathrm{mult}(\mathbf{1}, \rho_0) = 0$.

In what follows we show that if we put

$$T_s(D, \rho) := \sum_{\substack{p \\ N(p) \le q^D}} \frac{\mathrm{tr}(\rho(p))}{N(p)^s},$$

it holds that

(3) $$T_s(D, \mathbf{1}) = \begin{cases} \log D + \gamma - \log 2 + o(1) & (s \in 1 + \frac{\pi i}{\log q}\mathbf{Z}) \\ \log \zeta_\Gamma(s) + o(1) & (\text{otherwise}), \end{cases}$$

where $\zeta_\Gamma(s) = L_\Gamma(s, \mathbf{1})$, and that

(4) $$T_s(D, \rho_0) = c(\rho_0) + o(1)$$

with some constant $c(\rho_0)$.

We first show (3). We compute

$$\log\left(\prod_{\substack{p \\ N(p) \le q^D}} (1 - N(p)^{-s})^{-1}\right) = \sum_{\substack{p \\ N(p) \le q^D}} \sum_{k=1}^{\infty} \frac{1}{k} N(p)^{-sk} = A(D) + B(D),$$

where we put

$$A(D) = \sum_{\substack{k,d \\ kd \le D}} \frac{1}{k} q^{-dsk} \sum_{\substack{p \\ N(p) = q^d}} 1,$$

$$B(D) = \sum_{\substack{k,d \\ d \le D, kd > D}} \frac{1}{k} q^{-dsk} \sum_{\substack{p \\ N(p) = q^d}} 1.$$

We evaluate $A(D)$ by putting $l = kd$ as follows:

$$A(D) = \sum_{l=1}^{D} \sum_{d|l} \frac{d}{l} q^{-ls} \sum_{\substack{p \\ N(p) = q^d}} 1.$$

We apply Lemma 4 to $\rho = 1$, $n = 1$. Then it follows from [1] or [5] that

$$\zeta_\Gamma(s) = \frac{1 - q^{1-2s}}{1 - q^{2-2s}}.$$

Namely, we have $J = 4$ and $\alpha_1 = \sqrt{q}$, $\alpha_2 = -\sqrt{q}$, $\alpha_3 = q$, $\alpha_4 = -q$ with $v_1 = v_2 = -1$, $v_3 = v_4 = 1$.

Lemma 4 asserts that

$$\sum_{d|l} \sum_{\substack{p \\ N(p) = q^d}} 1 = q^l + (-q)^l - q^{\frac{l}{2}} - (-q^{\frac{1}{2}})^l.$$

Then

$$A(D) = \sum_{l=1}^{D} \frac{q^{-ls}}{l} (q^l + (-q)^l - q^{\frac{l}{2}} - (-q)^{\frac{l}{2}})$$
$$= \sum_{l=1}^{D} \frac{1}{l} (q^{l(1-s)}(1 + (-1)^l) + q^{\frac{l}{2}(\frac{1}{2}-s)}(1 + (-1)^{\frac{l}{2}})).$$

The second half of the sum tends to

$$\log(1 - q^{\frac{1}{2}-s})(1 + q^{\frac{1}{2}-s})$$

as $D \to \infty$. The first half behaves like

$$\sum_{1 \le k \le \frac{D}{2}} \frac{1}{k} q^{2k(1-s)} = \begin{cases} \log \frac{D}{2} + \gamma + o(1) & (s \in 1 + \frac{\pi i}{\log q}\mathbf{Z}) \\ \log(1 - q^{2(1-s)})^{-1} + o(1) & (\text{otherwise}) \end{cases}$$

as $D \to \infty$, since

$$1 + (-1)^l = \begin{cases} 2 & (l = 2k \in \mathbf{Z}) \\ 0 & (\text{otherwise}). \end{cases}$$

For $s \in 1 + \frac{\pi i}{\log q}\mathbf{Z}$, we reach

$$A(D) = \log D + \gamma - \log 2 + \log\left(1 - \frac{1}{q}\right) + o(1).$$

Whereas if $s \notin 1 + \frac{\pi i}{\log q}\mathbf{Z}$ with $\mathrm{Re}(s) = 1$, we have

$$A(D) = \log \zeta_\Gamma(s) + o(1) \qquad (D \to \infty).$$

We next deal with $B(D)$. It is easy to prove $B(D) \to 0$ as $D \to 0$ by the same way as Theorem 5. Indeed

$$B(D) = \sum_{k=2}^{\infty} \sum_{\substack{p \\ q^{\frac{D}{k}} < N(p) \le q^D}} \frac{N(p)^{-sk}}{k}$$

is a partial sum of an absolutely convergent series restricted to classes p with $N(p)$ sufficiently large.

Therefore we have

$$\log \prod_{\substack{p \\ N(p)\le q^D}} (1-N(p)^{-s})^{-1} = \begin{cases} \log\frac{D}{2}+\gamma+\log(1-q^{-1})+o(1) & (s\in 1+\frac{\pi i}{\log q}\mathbf{Z}) \\ \log\zeta_\Gamma(s)+o(1) & (\text{otherwise}) \end{cases}$$

as $D\to\infty$.

On the other hand, it is also computed as

$$\begin{aligned}\log \sum_{\substack{p \\ N(p)\le q^D}} (1-N(p)^{-s})^{-1} &= \sum_{\substack{p \\ N(p)\le q^D}} \sum_{k=1}^{\infty} \frac{N(p)^{-sk}}{k} \\ &= \sum_{\substack{p \\ N(p)\le q^D}} \frac{1}{N(p)^s} + \sum_{\substack{p \\ N(p)\le q^D}} \sum_{k=2}^{\infty} \frac{N(p)^{-sk}}{k} \\ &= T_s(D,\mathbf{1}) + \sum_{\substack{p \\ N(p)\le q^D}} (\log(1-N(p)^{-s})^{-1} - N(p)^{-s}) \\ &= T_s(D,\mathbf{1}) + \sum_{p}\sum_{k=2}^{\infty} \frac{N(p)^{-sk}}{k} + o(1) \quad (D\to\infty).\end{aligned}$$

Hence we obtain (3). It remains to prove (4). We apply Theorem 5 to $\rho=\rho_0$ and $x=q^D$. Then, since $\mathrm{Re}(s)=1$, we have as $D\to\infty$ that

$$\log L_{\Gamma,q^D}(s,\rho_0) = \log L_\Gamma(s,\rho_0)+o(1).$$

On the other hand, putting $n=\dim\rho_0$,

$$\begin{aligned}&\log L_{\Gamma,q^D}(s,\rho_0) \\ &= \log \prod_{\substack{p \\ N(p)\le q^D}} \det(I_n-\rho_0(p)N(p)^{-s})^{-1} \\ &= \sum_{\substack{p \\ N(p)\le q^D}} \sum_{k=1}^{\infty} \frac{\mathrm{tr}\,\rho_0(p)^k}{k} N(p)^{-sk} \\ &= T_s(D,\rho_0) + \sum_{\substack{p \\ N(p)\le q^D}} \sum_{k=2}^{\infty} \frac{\mathrm{tr}\,\rho_0(p)^k}{k} N(p)^{-sk} \\ &= T_s(D,\rho_0) + \sum_{\substack{p \\ \text{all}}} \sum_{k=2}^{\infty} \frac{\mathrm{tr}\,\rho_0(p)^k}{k} N(p)^{-sk} + o(1).\end{aligned}$$

Therefore, we conclude

$$\begin{aligned}&\mathrm{T}_s(D,\rho_0) \\ &= \log L_\Gamma(s,\rho_0) - \sum_{\substack{p \\ \text{all}}} \sum_{k=2}^{\infty} \frac{\mathrm{tr}\,\rho_0(p)^k}{k} N(p)^{-sk} + o(1).\end{aligned}$$

The result follows with

$$c(\rho_0) = \log L_\Gamma(s,\rho_0) - \sum_{\substack{p \\ \text{all}}} \sum_{k=2}^{\infty} \frac{\mathrm{tr}\,\rho_0(p)^k}{k} N(p)^{-sk}.$$

□

Theorem 7. *Assume* $\mathrm{Re}(s)=\frac{1}{2}$ *and* $L_\Gamma(s,\rho)\neq 0$. *If* $L_\Gamma(s,\rho)$ *satisfies Assumptions* 1 *and* 2, *then*

$$\lim_{x\to\infty} L_{\Gamma,x}(s,\rho) = L_\Gamma(s,\rho)\times \begin{cases} \sqrt{2}^c & (s\in\frac{1}{2}+\frac{\pi i}{\log q}\mathbf{Z}) \\ 1 & (\textit{otherwise}) \end{cases}$$

where c *is a constant depending on* ρ, *which is explicitly written by*

$$c = \mathrm{mult}(\mathbf{1},\mathrm{Sym}^2(\rho)) - \mathrm{mult}(\mathbf{1},\Lambda^2(\rho)).$$

Proof. As in the proof of Theorem 5, we decompose $\log L_{\Gamma,x}(s,\rho)$ into $A(x)+B(x)$, where $A(x)$ and $B(x)$ are given in the proof of Theorem 5. The identity $\lim_{x\to\infty} A(x) = \log L_\Gamma(s,\rho)$ is valid for $\mathrm{Re}(s)>0$.

We first deal with the partial sum for $k=2$ in $B(x)$:

$$\tilde{B}(x) = \sum_{\substack{p \\ x^{\frac{1}{2}}<N(p)\le x}} \frac{N(p)^{-2s}}{2} \sum_{i=1}^{n} \lambda_{p,i}^2.$$

We use the fact that

$$\sum_{i=1}^{n} \lambda_{p,i}^2 = \mathrm{tr}(\rho(p)^2) = \mathrm{tr}(\mathrm{Sym}^2(\rho)) - \mathrm{tr}(\Lambda^2(\rho)).$$

For each representation of $\mathrm{Sym}^2(\rho)$ and $\Lambda^2(\rho)$, we use Lemma 6 (Mertens' theorem). Then putting $x=q^D$ and $c=\mathrm{mult}(\mathbf{1},\mathrm{Sym}^2(\rho))-\mathrm{mult}(\mathbf{1},\Lambda^2(\rho))$, we compute

$$\begin{aligned}&\tilde{B}(q^D) \\ &= \sum_{\substack{p \\ q^{\frac{D}{2}}<N(p)\le q^D}} \frac{\mathrm{tr}(\mathrm{Sym}^2(\rho))-\mathrm{tr}(\Lambda^2(\rho))}{2} N(p)^{-2s} \\ &= \frac{1}{2}\left(T_{2s}(D,\mathrm{Sym}^2(\rho)) - T_{2s}\left(\frac{D}{2},\mathrm{Sym}^2(\rho)\right)\right. \\ &\quad \left. - T_{2s}(D,\lambda^2(\rho)) + T_{2s}\left(\frac{D}{2},\Lambda^2(\rho)\right)\right) \\ &= \begin{cases} \frac{c}{2}(\log D - \log\frac{D}{2}) + o(1) & (s\in\frac{1}{2}+\frac{\pi i}{2\log q}\mathbf{Z}) \\ o(1) & (\text{otherwise}) \end{cases} \\ &= \begin{cases} c\log\sqrt{2} + o(1) & (s\in\frac{1}{2}+\frac{\pi i}{2\log q}\mathbf{Z}) \\ o(1) & (\text{otherwise}) \end{cases}\end{aligned}$$

as $D\to\infty$.

The remaining sum over $k \geq 3$ tends to $o(1)$ as $D \to \infty$ by the same calculations as in the proof of Theorem 5. Hence we proved Theorem 7. □

Example 1. Let q be odd, $A \in \mathbf{F}_q[T]$, and $\Gamma(A)$ be the principal congruence subgroup of Γ with level A. Denote by ρ_0 the component of the regular representation ρ of the group $\Gamma/\Gamma(A)$ such that $\rho = \mathbf{1} \oplus \rho_0$. Then we easily calculate by using the formula in [8] that

$$L_\Gamma(s, \rho) = \frac{(1 - q^{1-2s})^{r-1}}{(1 - q^{-2s})^{\chi}} \det(T_{\Gamma(A)}, s)^{-1},$$

where $\chi = \mathrm{vol}(\Gamma \backslash PGL(2, \mathbf{F}_q((T^{-1}))))(q-1)/2$, $r = \frac{1}{2}\,\mathrm{tr}(I_n - \Phi(\frac{1}{2}))$ with Φ the scattering matrix, and $\det(T_{\Gamma(A)}, s)$ is the determinant of the Laplacian consisting of both discrete and continuous spectra, which turns to be a rational function in q^{-s}. Then it satisfies Assumptions 1 and 2, and we obtain the deep Riemann hypothesis for the Selberg zeta functions for any principal congruence subgroups.

Example 2. If $q = 3$ and ρ is the nontrivial 1-dimensional character of

$$\Gamma/PSL(2, \mathbf{F}_3[T]) \cong \mathbf{Z}/2\mathbf{Z},$$

we have $n = 1$ and $\lambda_{p,1}^2 = 1$. The Selberg zeta function is given by [6] as

$$L_\Gamma(s, \rho) = \frac{1 - 3^{1-2s}}{1 + 3^{1-2s}},$$

which satisfies Assumptions 1 and 2. Thus Theorem 7 holds with $c = 1$.

If we consider more general q $(q \geq 5)$, we compute from [6] that

$$L_\Gamma(s, \rho) = \frac{1 - q^{1-2s}}{1 + (q-2)q^{1-2s}},$$

which does not satisfy Assumption 2, because we have $|\alpha_j| = \sqrt{(q-2)q} > \sqrt{q}$ for some j. Indeed $L_\Gamma(s, \rho)$ has a pole at $s = \frac{1}{2}(1 + \log_q(q-2)) < 1$, and our current method does not enable us to extend the convergence region passing the pole. In such case, however, we can obtain some asymptotic behavior of the Euler product by the technique of Kimura–Koyama–Kurokawa [4] and Akatsuka [2], which will be treated in the forthcoming paper.

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