The exponential calculus of pseudodifferential operators of minimum type. I

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Abstract: In this paper we show that for each formal symbol p of minimum type there is a formal symbol q of minimum type satisfying $\exp(: p :) =: \exp(q) :$.

Key words: Exponential calculus; minimum type; symbols.

1. Introduction. Sato, M., T. Kawai and M. Kashiwara won epoch-making results in the theory of transformation of the system of linear partial (pseudo) differential equations by using pseudodifferential operators of infinite order (cf. [5]). T. Aoki accomplished exponential calculus of analytic pseudodifferential operators (see [1]). We consider the case of a kind of the direct product structure (see [2–4]). That is, we consider calculus of analytic pseudodifferential operators of product type and of minimum type and generalize the theory of T. Aoki in some sense. The study of minimum type is connected with exponential calculus of positive definite operators of infinite order which have deep relation to the energy method in the hyperfunction theory (see [2]). We naturally define an exponential function of a pseudodifferential operator of minimum type as a pseudodifferential operator of product type (see [4]). In this paper we show that for each formal symbol p of minimum type there is a formal symbol q of minimum type satisfying $\exp(: p :) =: \exp(q) :$. That is, the exponential function of an operator of minimum type has an exponential symbol.

2. Symbols of minimum type. Let $X \subset \mathbf{C}^n$ and $Y \subset \mathbf{C}^m$ be domains. Then, the cotangent bundles T^*X and T^*Y are identified with $X \times \mathbf{C}^n$ and $Y \times \mathbf{C}^m$, respectively. We set

$$S^*X := (T^*X - X)/\mathbf{R}^+, \ S^*Y := (T^*Y - Y)/\mathbf{R}^+$$

and define the mapping γ as

$$\gamma\colon \overset{\circ\circ}{T^*}(X \times Y) \ni (x, y; \xi, \eta) \longmapsto \left(x; \frac{\xi}{|\xi|}\right) \times \left(y; \frac{\eta}{|\eta|}\right) \in S^*X \times S^*Y,$$

where $\overset{\circ\circ}{T^*}(X \times Y) := T^*(X \times Y) \backslash \{(T^*X \times Y) \cup (X \times T^*Y)\}$. For $d_1, d_2 > 0$ and an open subset U of $S^*X \times S^*Y$, we use the notation

$$\gamma^{-1}(U; d_1, d_2) := \gamma^{-1}(U) \cap \{|\xi| > d_1, |\eta| > d_2\}.$$

Hereafter we write (x, ξ, y, η) for coordinates $(x, y; \xi, \eta)$.

Let K be a compact subset of $S^*X \times S^*Y$.

Definition 2.1. A formal series $\sum_{j,k=0}^{\infty} P_{j,k}(x, \xi, y, \eta)$ is called a formal symbol of product type on K if the following hold.

(1) There are some constants $d > 0$, $0 < A < 1$, and an open set $U \supset K$ in $S^*X \times S^*Y$ such that $P_{j,k}$ is holomorphic in $\gamma^{-1}(U; (j+1)d, (k+1)d)$ for each $j, k \geq 0$.

(2) For each $\varepsilon > 0$, there is some constant $C_\varepsilon > 0$ such that

$$|P_{j,k}(x, \xi, y, \eta)| \leq C_\varepsilon A^{j+k} e^{\varepsilon(|\xi|+|\eta|)} \tag{2.1}$$

on $\gamma^{-1}(U; (j+1)d, (k+1)d)$ for each $j, k \geq 0$.

We denote by $\widehat{S}(K)$ the set of such formal symbols on K and often write a formal power series $\sum_{j,k=0}^{\infty} t_1^j t_2^k P_{j,k}(x, \xi, y, \eta)$ with indeterminates t_1 and t_2 instead of $\sum_{j,k=0}^{\infty} P_{j,k}(x, \xi, y, \eta)$.

Definition 2.2. We denote by $\widehat{R}(K)$ the set of all $P(t_1, t_2; x, \xi, y, \eta) := \sum_{j,k=0}^{\infty} t_1^j t_2^k P_{j,k}(x, \xi, y, \eta)$ in $\widehat{S}(K)$ such that there are some constants $d > 0$, $0 < A < 1$, and an open set $U \supset K$ in $S^*X \times S^*Y$ satisfying the following: for each $\varepsilon > 0$, there is some constant $C_\varepsilon > 0$ such that

$$\left| \sum_{\substack{0 \leq j \leq s \\ 0 \leq k \leq t}} P_{j,k}(x, \xi, y, \eta) \right| \leq C_\varepsilon A^{\min\{s,t\}} e^{\varepsilon(|\xi|+|\eta|)} \tag{2.2}$$

on $\gamma^{-1}(U; (s+1)d, (t+1)d)$ for each $s, t \geq 0$.

We call an element of $\widehat{R}(K)$ a formal symbol of zero class. $\widehat{S}(K)/\widehat{R}(K)$ becomes a commutative ring (see [3]).

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Definition 2.3. We call an element in the ring $\widehat{S}(K)/\widehat{R}(K)$ a pseudo-differential operator of product type on K. We write $:\sum P_{j,k}:$ for the associated pseudodifferential operator of product type on K using an element $\sum P_{j,k}$ in $\widehat{S}(K)$.

Definition 2.4. A function $\Lambda : \mathbf{R}_{>0} \longrightarrow \mathbf{R}_{>0}$ is said to be infra-linear if the following conditions hold:

(a) Λ is continuous;

(b) for each $\alpha > 1$, $\Lambda(\alpha t) \le \alpha\Lambda(t)$ on $(0, \infty)$;

(c) Λ is increasing;

(d) $\displaystyle\lim_{t\to\infty} \frac{\Lambda(t)}{t} = 0$.

Hereafter we fix two infra-linear functions Λ, Λ^* and put

$$\tilde{\Lambda}(\xi,\eta) = \min\{\Lambda(|\xi|), \Lambda^*(|\eta|)\}.$$

Definition 2.5. $\sum P_{j,k}$ in $\widehat{S}(K)$ is called a formal symbol of minimum type of growth order (Λ, Λ^*), or simply of $\widetilde{\Lambda}$-type on K if there exist some constants $C > 0$, $d > 0$, $0 < A < 1$, and an open set $U \supset K$ in $S^*X \times S^*Y$ satisfying the following: (1) $P_{j,k}$ is holomorphic in $\gamma^{-1}(U; (j+1)d, (k+1)d)$ for each $j, k \ge 0$,
(2) the inequality

$$|P_{j,k}(x,\xi,y,\eta)| \le CA^{j+k} \min\{\Lambda(|\xi|), \Lambda^*(|\eta|)\}$$

holds on $\gamma^{-1}(U; (j+1)d, (k+1)d)$ for each $j, k \ge 0$.

Remark 2.6. If $\sum P_{j,k}$ is a formal symbol of minimum type on K, $e^{\sum P_{j,k}}$ is a formal symbol of product type on K.

From now on, we use the notation t, t^*, x^*, ξ^* instead of t_1, t_2, y, η. Furthermore, $P\begin{pmatrix} t; & x, & \xi \\ t^*; & x^*, & \xi^* \end{pmatrix}$ stands for a formal symbol of product type

$$P(t,t^*;x,\xi,x^*,\xi^*) = \sum_{j,j^*\ge 0} t^j t^{*j^*} p_{j,j^*}(x,\xi,x^*,\xi^*)$$

3. Exponential function of operators. Suppose that

$$p\begin{pmatrix} t; & x, & \xi \\ t^*; & x^*, & \xi^* \end{pmatrix} = \sum_{j,j^*\ge 0} t^j t^{*j^*} p_{j,j^*}\begin{pmatrix} x, & \xi \\ x^*, & \xi^* \end{pmatrix}$$

is a formal symbol of $\widetilde{\Lambda}$-type on K.

Then we shall define the operator $\exp\left(s : p\begin{pmatrix} t; & x, & \xi \\ t^*; & x^*, & \xi^* \end{pmatrix} :\right)$ for each complex number s.

To define the operator $\exp(s : p :)$ $(s \in \mathbf{C})$, we set the sequence $\{p^{(l)}\}(l \ge 0)$ as follows by introducing copies t_1, t_1^*, t_2, t_2^* of t, t^*:

$$(3.1)\quad p^{(0)}\begin{pmatrix} t_1, & t_2; & x, & \xi \\ t_1^*, & t_2^*; & x^*, & \xi^* \end{pmatrix} := 1,$$

$$(3.2)\quad p^{(l+1)}\begin{pmatrix} t_1, & t_2; & x, & \xi \\ t_1^*, & t_2^*; & x^*, & \xi^* \end{pmatrix} := \exp(t_2\langle\partial_\xi, \partial_y\rangle + t_2^*\langle\partial_{\xi^*}, \partial_{y^*}\rangle) p\begin{pmatrix} t_1; & x, & \xi \\ t_1^*; & x^*, & \xi^* \end{pmatrix} \times p^{(l)}\begin{pmatrix} t_1, & t_2; & y, & \eta \\ t_1^*, & t_2^*; & y^*, & \eta^* \end{pmatrix}\Bigg|_{\begin{pmatrix} y, & \eta \\ y^*, & \eta^* \end{pmatrix} = \begin{pmatrix} x, & \xi \\ x^*, & \xi^* \end{pmatrix}}.$$

By the definition of $p^{(l)}$, we obtain

$$: p^{(l)}\begin{pmatrix} t, & t; & x, & \xi \\ t^*, & t^*; & x^*, & \xi^* \end{pmatrix} : = (: p :)^l.$$

We put

$$p^{(l)}\begin{pmatrix} t_1, & t_2; & x, & \xi \\ t_1^*, & t_2^*; & x^*, & \xi^* \end{pmatrix} := \sum_{\substack{j,k\ge 0 \\ j^*,k^*\ge 0}} t_1^j t_2^k t_1^{*j^*} t_2^{*k^*} p^{(l)}_{(j,\,k)(j^*,\,k^*)}\begin{pmatrix} x, & \xi \\ x^*, & \xi^* \end{pmatrix}.$$

Then by (3.1) and (3.2), we obtain the following recursive formulas:

$$p^{(0)}_{(j,\,k)(j^*,\,k^*)} = \begin{cases} 1, & \text{if } (j,\ k,\ j^*,\ k^*) = (0,\ 0,\ 0,\ 0) \\ 0, & \text{otherwise} \end{cases},$$

$$p^{(l+1)}_{(j,\,k)(j^*,\,k^*)}\begin{pmatrix} x, & \xi \\ x^*, & \xi^* \end{pmatrix} = \sum_{\substack{i+\mu=j,\ |\alpha|+\nu=k \\ i^*+\mu^*=j^*, |\alpha^*|+\nu^*=k^*}} \frac{1}{\alpha!}\frac{1}{\alpha^*!}\partial_\xi^\alpha \partial_{\xi^*}^{\alpha^*} p_{i,i^*}\begin{pmatrix} x, & \xi \\ x^*, & \xi^* \end{pmatrix} \times \partial_x^\alpha \partial_{x^*}^{\alpha^*} p^{(l)}_{(\mu,\nu)(\mu^*,\nu^*)}\begin{pmatrix} x, & \xi \\ x^*, & \xi^* \end{pmatrix}.$$

It is clear that $p^{(l)}_{(j,k)(j^*,k^*)}$ is holomorphic on $\gamma^{-1}(U; (j+1)d, (j^*+1)d)$ for $j, j^* \ge 0$. On the other hand, we put

$$E\begin{pmatrix} t_1, & t_2; & s, & x, & \xi \\ t_1^*, & t_2^*; & & x^*, & \xi^* \end{pmatrix} = \sum_{l=0}^{\infty} \frac{s^l}{l!} p^{(l)}\begin{pmatrix} t_1, & t_2; & x, & \xi \\ t_1^*, & t_2^*; & x^*, & \xi^* \end{pmatrix}.$$

Then we obtain the following proposition and theorem in [4].

Proposition 3.1. *For each $s \in \mathbf{C}$, the formal series $E\begin{pmatrix} t, & t; & s, & x, & \xi \\ t^*, & t^*; & & x^*, & \xi^* \end{pmatrix}$ is a formal* *symbol of product type of growth order* $e^{\tilde{\Lambda}}$.

Therefore we can define $\exp(s:p:)$ as

$$\exp(s:p:) := :E\begin{pmatrix} t, & t; & s, & x, & \xi \\ t^*, & t^*; & & x^*, & \xi^* \end{pmatrix}:.$$

Theorem 3.2. *Suppose that* p *is a formal symbol of* $\widetilde{\Lambda}$*-type on* K. *Then* $\exp(s:p:)$ *satisfies the following*

$$\begin{cases} \partial_s \exp(s:p:) =: p : \exp(s:p:), \\ \exp(s:p:)|_{s=0} = 1. \end{cases} \tag{3.3}$$

Moreover, the following exponential law holds:

$$\exp(s_1:p:)\ \exp(s_2:p:) = \exp((s_1+s_2):p:).$$

In particular, $\exp(-s:p:)$ *is the inverse operator of* $\exp(s:p:)$.

4. Main result. Let $p\begin{pmatrix} t; & x, & \xi \\ t^*; & x^*, & \xi^* \end{pmatrix}$ be a formal symbol of $\tilde{\Lambda}$-type on K. Then, we see that $\exp(s:p:)$ ($s \in \mathbf{C}$) defined by

$$\exp(s:p:) =: E\begin{pmatrix} t_1, & t_2; & s, & x, & \xi \\ t_1^*, & t_2^*; & & x^*, & \xi^* \end{pmatrix}\Bigg|_{\begin{pmatrix} t_1=t_2=t \\ t_1^*=t_2^*=t^* \end{pmatrix}}:$$

is a formal symbol of product type of growth order $e^{\tilde{\Lambda}}$. In this section, we construct $q\begin{pmatrix} t; & s, & x, & \xi \\ t^*; & & x^*, & \xi^* \end{pmatrix}$ as a formal symbol of $\tilde{\Lambda}$-type such that for each $s \in \mathbf{C}$

$$\exp(s:p:) =: \exp\left(q\begin{pmatrix} t; & s, & x, & \xi \\ t^*; & & x^*, & \xi^* \end{pmatrix}\right):.$$

We put

$$q := \sum_{i,i^*\geq 0} t^i t^{*i^*} q_{i,i^*}\begin{pmatrix} s, & x, & \xi \\ & x^*, & \xi^* \end{pmatrix} = \sum_{i,i^*\geq 0} t^i t^{*i^*} \sum_{l=1}^{\infty} s^l q^{(l)}_{i,i^*}\begin{pmatrix} x, & \xi \\ x^*, & \xi^* \end{pmatrix}.$$

Due to the idea in [1] we construct q as the solution of the following equation:

$$\begin{cases} \partial_s : \exp(q) :=: p :: \exp(q):, \\ : \exp q\begin{pmatrix} t; & 0, & x, & \xi \\ t^*; & & x^*, & \xi^* \end{pmatrix} := 1 \end{cases},$$

where

$$: p :: \exp(q) : =: \exp\big(t\partial_\xi\cdot\partial_y + t^*\partial_{\xi^*}\cdot\partial_{y^*}\big)\left[p\begin{pmatrix} t; & x, & \xi \\ t^*; & x^*, & \xi^* \end{pmatrix} \times \exp\left(q\begin{pmatrix} t; & s, & y, & \eta \\ t^*; & & y^*, & \eta^* \end{pmatrix}\right)\right]\Bigg|_{\substack{y=x,\ \eta=\xi \\ y^*=x^*, \eta^*=\xi^*}}:.$$

Hence we introduce another formal series ψ and q by the equation:

$$\begin{aligned} &\exp\big(t_3t_2\partial_\xi\cdot\partial_y + t_3^*t_2^*\partial_{\xi^*}\cdot\partial_{y^*}\big)\left[p\begin{pmatrix} t_1; & x, & \xi \\ t_1^*; & x^*, & \xi^* \end{pmatrix} \times \exp\left(q\begin{pmatrix} t_1, & t_2; & s, & y, & \eta \\ t_1^*, & t_2^*; & & y^*, & \eta^* \end{pmatrix}\right)\right] \\ &= \psi\begin{pmatrix} t_1, & t_2, & t_3; & s, & x, & \xi, & y, & \eta \\ t_1^*, & t_2^*, & t_3^*; & & x^*, & \xi^*, & y^*, & \eta^* \end{pmatrix} \times \exp\left(q\begin{pmatrix} t_1, & t_2; & s, & y, & \eta \\ t_1^*, & t_2^*; & & y^*, & \eta^* \end{pmatrix}\right). \end{aligned} \tag{4.1}$$

Then, it is easy to see that ψ and $\exp q$ in the right side of (4.1) need to satisfy the following equalities:

$$\begin{cases} \partial_{t_3}\psi = e^{-q}\partial_{t_3}(\psi e^q) = e^{-q}t_2\partial_\xi\cdot\partial_y(\psi e^q) \\ \qquad = t_2(\partial_\xi\cdot\partial_y\psi + \partial_y q\cdot\partial_\xi\psi), \\ \partial_{t_3^*}\psi = e^{-q}\partial_{t_3^*}(\psi e^q) = e^{-q}t_2^*\partial_{\xi^*}\cdot\partial_{y^*}(\psi e^q) \\ \qquad = t_2^*(\partial_{\xi^*}\cdot\partial_{y^*}\psi + \partial_{y^*}q\cdot\partial_{\xi^*}\psi), \end{cases}$$

$$\begin{aligned} : e^q\partial_s q\begin{pmatrix} t, & t; & s, & x, & \xi \\ t^*, & t^*; & & x^*, & \xi^* \end{pmatrix} &:= \partial_s(: e^q :) \\ &=: p :: e^q : \\ &=: e^q\psi\begin{pmatrix} t, & t, & 1; & s, & x, & \xi, & x, & \xi \\ t^*, & t^*, & 1; & & x^*, & \xi^*, & x^*, & \xi^* \end{pmatrix}:. \end{aligned}$$

Hence by putting

$$q\begin{pmatrix} t_1, & t_2; & s, & x, & \xi \\ t_1^*, & t_2^*; & & x^*, & \xi^* \end{pmatrix} = \sum_{l,i,j,i^*,j^*} s^l t_1^i t_2^j t_1^{*i^*} t_2^{*j^*} q^{(l)}_{(i,j)(i^*,j^*)}\begin{pmatrix} x, & \xi \\ x^*, & \xi^* \end{pmatrix}, \tag{4.2}$$

$$\psi = \sum_{l,i,j,k,i^*,j^*,k^*} s^l t_1^i t_2^j t_3^k t_1^{*i^*} t_2^{*j^*} t_3^{*k^*} \psi^{(l)}_{(i,j,k)(i^*,j^*,k^*)}\begin{pmatrix} x, & \xi, & y, & \eta \\ x^*, & \xi^*, & y^*, & \eta^* \end{pmatrix}, \tag{4.3}$$

we have the following recursive formulas:

$$q^{(l+1)}_{(i,j)(i^*,j^*)}\begin{pmatrix} x, & \xi \\ x^*, & \xi^* \end{pmatrix} = \frac{1}{l+1}\sum_{\substack{0\leq k\leq j \\ 0\leq k^*\leq j^*}} \psi^{(l)}_{(i,j,k)(i^*,j^*,k^*)} \tag{4.4}$$

$$\times \begin{pmatrix} x, & \xi, & x, & \xi \\ x^*, & \xi^*, & x^*, & \xi^* \end{pmatrix},$$

$$(4.5)\ (k+1)\psi^{(l)}_{(i,j,k+1)(i^*,j^*,k^*)}\begin{pmatrix} x, & \xi, & y, & \eta \\ x^*, & \xi^*, & y^*, & \eta^* \end{pmatrix}$$
$$= \partial_\xi \cdot \partial_y \psi^{(l)}_{(i,j-1,k)(i^*,j^*,k^*)}\begin{pmatrix} x, & \xi, & y, & \eta \\ x^*, & \xi^*, & y^*, & \eta^* \end{pmatrix}$$
$$+ \sum{}'' \sum_{l'+l''=l} \partial_\xi \psi^{(l')}_{(i',j',k)(i'^*,j'^*,k^*)}$$
$$\times \begin{pmatrix} x, & \xi, & y, & \eta \\ x^*, & \xi^*, & y^*, & \eta^* \end{pmatrix}$$
$$\cdot \partial_y q^{(l'')}_{(i'',j'')(i''^*,j''^*)}\begin{pmatrix} y, & \eta \\ y^*, & \eta^* \end{pmatrix},$$

$$(4.6)\ (k^*+1)\psi^{(l)}_{(i,j,k)(i^*,j^*,k^*+1)}\begin{pmatrix} x, & \xi, & y, & \eta \\ x^*, & \xi^*, & y^*, & \eta^* \end{pmatrix}$$
$$= \partial_{\xi^*} \cdot \partial_{y^*} \psi^{(l)}_{(i,j,k)(i^*,j^*-1,k^*)}\begin{pmatrix} x, & \xi, & y, & \eta \\ x^*, & \xi^*, & y^*, & \eta^* \end{pmatrix}$$
$$+ \sum{}''^* \sum_{l'+l''=l}$$
$$\cdot \partial_{\xi^*} \psi^{(l')}_{(i',j',k)(i'^*,j'^*,k^*)}\begin{pmatrix} x, & \xi, & y, & \eta \\ x^*, & \xi^*, & y^*, & \eta^* \end{pmatrix}$$
$$\cdot \partial_{y^*} q^{l''}_{(i'',j'')(i''^*,j''^*)}\begin{pmatrix} y, & \eta \\ y^*, & \eta^* \end{pmatrix},$$

where $\sum''$ is the sum ranging over $i'+i''=i$, $j'+j''=j-1$, $i'^*+i''^*=i^*$, $j'^*+j''^*=j^*$ and $\sum''^*$ is the sum ranging over $i'+i''=i$, $j'+j''=j$, $i'^*+i''^*=i^*$, $j'^*+j''^*=j^*-1$. We see that

$$q^{(l)}_{i,i^*}\begin{pmatrix} x, & \xi \\ x^*, & \xi^* \end{pmatrix} = \sum_{\substack{0\le j\le i \\ 0\le j^*\le i^*}} q^{(l)}_{(i-j,j)(i^*-j^*,j^*)}\begin{pmatrix} x, & \xi \\ x^*, & \xi^* \end{pmatrix}.$$

Lemma 4.1. *We have the following initial and boundary conditions:*

$$(4.7)\qquad \psi^{(l)}_{(i,j,0)(i^*,j^*,0)}\begin{pmatrix} x, & \xi, & y, & \eta \\ x^*, & \xi^*, & y^*, & \eta^* \end{pmatrix} = \delta_{l,0}\delta_{j,0}\delta_{j^*,0} \cdot p_{i,i^*}\begin{pmatrix} x, & \xi \\ x^*, & \xi^* \end{pmatrix},$$

$$(4.8)\qquad \psi^{(l)}_{(i,j,k)(i^*,j^*,k^*)}\begin{pmatrix} x, & \xi, & y, & \eta \\ x^*, & \xi^*, & y^*, & \eta^* \end{pmatrix} = 0$$
$$(j<l \text{ or } j<k \textit{ or } j^*<l \textit{ or } j^*<k^*),$$

$$(4.9)\qquad q^{(l)}_{(i,j)(i^*,j^*)} = 0 \textit{ if } l>j+1 \textit{ or } l>j^*+1.$$

Proof. The first equation directly follows from (4.1). Further, the second equation is also clear from (4.1) when $j<k$, or $j^*<k^*$. On the other hand, using the recursive formulas above, we can prove the second equation for $j<l$ or $j^*<l$, and the third equation for $j+1<l$ or $j^*+1<l$ simultaneously by double mathematical induction on l, and $k+k^*$, that is, the main induction on l and the supplementary induction on $k+k^*$. □

Therefore, we have the following:

$$(4.10)\qquad q_{i,i^*}\begin{pmatrix} s, & x, & \xi \\ & x^*, & \xi^* \end{pmatrix} = p_{i,i^*}\begin{pmatrix} x, & \xi \\ x^*, & \xi^* \end{pmatrix} + \sum_{\substack{0\le j\le i \\ 0\le j^*\le i^* \\ (j,j^*)\ne(0,0)}} \sum_{l=1}^{\min\{j,j^*\}+1} s^l q^{(l)}_{(i-j,j)(i^*-j^*,j^*)} \times \begin{pmatrix} x, & \xi \\ x^*, & \xi^* \end{pmatrix}.$$

Theorem 4.2. *For each $s\in\mathbf{C}$, the formal series q is a formal symbol of $\tilde\Lambda$-type on K satisfying the following:*

$$:\exp q\begin{pmatrix} t, & t; & s, & x, & \xi \\ t^*, & t^*; & & x^*, & \xi^* \end{pmatrix} := \exp(s:p:).$$

5. Outline of the proof of Theorem 4.2. Suppose that p satisfies the following: there exist some constants δ', $d>0$, $0<A<1$, and an open set $U(\supset K)$ in $S^*X\times S^*Y$ such that

$$\left| p_{i,i^*}\begin{pmatrix} x, & \xi \\ x^*, & \xi^* \end{pmatrix}\right| \le (i+1)^{-2}(i^*+1)^{-2}A^{i+i^*}\tilde\Lambda(\xi,\xi^*)$$

on $\gamma^{-1}(U;(i+1)d_{\delta'},(i^*+1)d_{\delta'})$, where $d_{\delta'} := d(1-\delta')$. We introduce some notations:

$$V := \gamma^{-1}(U)\times\gamma^{-1}(U),$$

$$Z^{\delta'} := \left\{ \begin{pmatrix} x, & \xi, & y, & \eta \\ x^*, & \xi^*, & y^*, & \eta^* \end{pmatrix} \in V; \right.$$
$$|\xi'-\xi|\le\delta'|\xi|, |y'-y|\le\delta',$$
$$|\xi^{*\prime}-\xi^*|\le\delta'|\xi^*|, |y^{*\prime}-y^*|\le\delta',$$
$$|\eta'-\eta|\le\delta'|\eta|, |x'-x|\le\delta',$$
$$|\eta^{*\prime}-\eta^*|\le\delta'|\eta^*|, |x^{*\prime}-x^*|\le\delta'$$
$$\left. \Longrightarrow \begin{pmatrix} x', & \xi', & y', & \eta' \\ x^{*\prime}, & \xi^{*\prime}, & y^{*\prime}, & \eta^{*\prime} \end{pmatrix} \in V \right\},$$

$$Z^{\delta'}_{(\epsilon_1,\epsilon_2)(\epsilon_1^*,\epsilon_2^*)} := \left\{ \begin{pmatrix} x', & \xi', & y', & \eta' \\ x^{*\prime}, & \xi^{*\prime}, & y^{*\prime}, & \eta^{*\prime} \end{pmatrix}; \right.$$
$$\exists \begin{pmatrix} x, & \xi, & y, & \eta \\ x^*, & \xi^*, & y^*, & \eta^* \end{pmatrix} \in Z^{\delta'} \text{ such that}$$
$$|\xi'-\xi|\le\epsilon_1|\xi|,\ |y'-y|\le\epsilon_2,$$
$$|\xi^{*\prime}-\xi^*|\le\epsilon_1^*|\xi^*|,\ |y^{*\prime}-y^*|\le\epsilon_2^*,$$

$$\left.\begin{aligned}&|\eta' - \eta| \le \epsilon_1|\eta|,\ |x' - x| \le \epsilon_2,\\ &|\eta^{*\prime} - \eta^*| \le \epsilon_1^*|\eta^*|,\ |x^{*\prime} - x^*| \le \epsilon_2^*\end{aligned}\right\}$$

$$Z^{\delta'}_{(\epsilon_1,\epsilon_2)(\epsilon_1^*,\epsilon_2^*)}(d_1, d_2; d_1^*, d_2^*) := Z^{\delta'}_{(\epsilon_1,\epsilon_2)(\epsilon_1^*,\epsilon_2^*)} \cap \{|\xi| \ge d_1,\ |\eta| \ge d_2,\ |\xi^*| \ge d_1^*,\ |\eta^*| \ge d_2^*\},$$

$$DZ^{\delta'}_{\epsilon,\epsilon^*}(d; d^*) := \left\{\begin{pmatrix} x, & \xi \\ x^*, & \xi^*\end{pmatrix};\ \begin{pmatrix} x, & \xi, & x, & \xi \\ x^*, & \xi^*, & x^*, & \xi^*\end{pmatrix} \in Z^{\delta'}_{(\epsilon,\epsilon)(\epsilon^*,\epsilon^*)}(d, d; d^*, d^*)\right\},$$

$$C_{j_1,\cdots,j_m} := \prod_{\nu=1}^{m} \frac{1}{(j_\nu + 1)^2},$$

where j_ν is a nonnegative integer for each natural number ν. Then, we can prove the theorem using the following lemma.

Lemma 5.1. *There exist* $N > 1$, $C > 1$, $\delta \in (0, \delta')$ *such that the following* (1) *and* (2) *hold:*
(1) *For any* ϵ_1, ϵ_2, ϵ_1^*, $\epsilon_2^* \in (0, \delta)$ *and any* $\epsilon_2' \in (0, \epsilon_2)$, ${\epsilon_2'}^* \in (0, \epsilon_2^*)$

$$(5.1)\quad \left|\psi^{(l)}_{(i,j,k)(i^*,j^*,k^*)}\begin{pmatrix} x, & \xi, & y, & \eta \\ x^*, & \xi^*, & y^*, & \eta^*\end{pmatrix}\right| \le \frac{1}{l!(l+1)^2}\tilde{\Lambda}(\xi,\xi^*)\tilde{\Lambda}(\eta,\eta^*)^l|\xi|^{-k}|\xi^*|^{-k^*} \times |\eta|^{-j+k}|\eta^*|^{-j^*+k^*} \times j!C_{ijk}[(\delta - \epsilon_1)^{-k}(\delta - \epsilon_2)^{-2j+k}]^N \times [A\delta^{2N}(\delta - \epsilon_2')^{-2N}]^i \times j^*!C_{i^*j^*k^*}[(\delta - \epsilon_1^*)^{-k^*}(\delta - \epsilon_2^*)^{-2j^*+k^*}]^N \times [A\delta^{2N}(\delta - {\epsilon_2'}^*)^{-2N}]^{i^*} \qquad (l \ge 0)$$

on $Z^{\delta'}_{(\epsilon_1,\epsilon_2)(\epsilon_1^*,\epsilon_2^*)}((i+1)d_{\epsilon_1}, (i+1)d_{\epsilon_2}; (i^*+1)d_{\epsilon_1^*}, (i^*+1)d_{\epsilon_2^*})$.

(2) *For any* $\epsilon' \in (0, \epsilon)$, $\epsilon'^* \in (0, \epsilon^*)$

$$(5.2)\quad \left|q^{(l+1)}_{(i,j)(i^*,j^*)}\begin{pmatrix} x, & \xi \\ x^*, & \xi^*\end{pmatrix}\right| \le \frac{C}{(l+1)!(l+2)^2}\tilde{\Lambda}(\xi,\xi^*)\tilde{\Lambda}(\xi,\xi^*)^l \times j!C_{ij}(\delta - \epsilon)^{-2jN}[A\delta^{2N}(\delta - \epsilon')^{-2N}]^i|\xi|^{-j} \times j^*!C_{i^*j^*}(\delta - \epsilon^*)^{-2j^*N} \times [A\delta^{2N}(\delta - \epsilon'^*)^{-2N}]^{i^*}|\xi^*|^{-j^*}$$

$(l \ge 0)$ *on* $DZ^{\delta'}_{\epsilon,\epsilon^*}((i+1)d_\epsilon; (i^*+1)d_{\epsilon^*})$.

Detailed proofs of the above lemma and Theorem 4.2 will be published elsewhere.

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References

[1] T. Aoki, Calcul exponentiel des opérateurs microdifférentiels d'ordre infini. II, Ann. Inst. Fourier (Grenoble) **36** (1986), no. 2, 143–165.
[2] K. Kataoka, Microlocal energy methods and pseudodifferential operators, Invent. Math. **81** (1985), no. 2, 305–340.
[3] C. H. Lee, Composition of pseudodifferential operators of product type, Proc. Jangjeon Math. Soc. **12** (2009), no. 3, 289–298.
[4] C. H. Lee, Exponential function of pseudodifferential operator of minimum type, Proc. Jangjeon Math. Soc. **14** (2011), no. 1, 149–160.
[5] M. Sato, T. Kawai and M. Kashiwara, Microfunctions and pseudo-differential equations, in *Hyperfunctions and pseudo-differential equations* (*Proc. Conf., Katata, 1971; dedicated to the memory of André Martineau*), 265–529, Lecture Notes in Math., 287, Springer, Berlin, 1973.