Deformations of the discrete Heisenberg group

By Severin BARMEIER

Instituto de Matemática, Estatística e Computação Científica, Universidade Estadual de Campinas, Rua Sérgio Buarque de Holanda, 651, Cidade Universitária "Zeferino Vaz", Campinas, SP, Brazil

(Communicated by Masaki KASHIWARA, M.J.A., March 12, 2013)

Abstract: We study deformations of the discrete Heisenberg group acting properly discontinuously on the Heisenberg group from the left and right and obtain a complete description of the deformation space.

Key words: Deformation; discrete group; properly discontinuous action; homogeneous space; Heisenberg group.

1. Introduction and statement of main result. We will be interested in deformations of the discrete Heisenberg group as a group acting properly discontinuously and cocompactly on a space X. The following defines our notion of *deformation*.

Definition 1.1 ([K93, K01, KN06]). Let G be a Lie group acting continuously on a locally compact space X and let $\Gamma \subset G$ be a discrete subgroup. Define the parameter space of deformations of Γ within G, acting properly discontinuously on the space X as

$$R(\Gamma, G; X) = \left\{ \phi\colon \Gamma \to G \;\middle|\; \begin{array}{l} \phi \text{ is injective,} \\ \phi(\Gamma) \text{ acts properly} \\ \text{discontinuously} \\ \text{and freely on } X \end{array} \right\}$$

and the deformation space as

$$\mathcal{T}(\Gamma, G; X) = R(\Gamma, G; X)/G,$$

where G acts on $R(\Gamma, G; X)$ by conjugation, so that $\mathcal{T}(\Gamma, G; X)$ is the space of non-trivial deformations.

There is a natural topology on the parameter space $R(\Gamma, G; X)$ as a subset of $\mathrm{Hom}(\Gamma, G)$ endowed with the compact open topology. We then consider the quotient topology on the deformation space $\mathcal{T}(\Gamma, G; X)$ ([K93, K01]).

If X is an irreducible Riemannian symmetric space G/K, Selberg–Weil rigidity ([W64]) states that $\mathcal{T} = \mathcal{T}(\Gamma, G; G/K)$ is discrete if and only if G is not locally isomorphic to $\mathrm{SL}_2\mathbf{R}$. An example of the failure of rigidity is when $G = \mathrm{PSL}_2\mathbf{R}$, Γ is the fundamental group of a Riemann surface of genus $g \geqslant 2$ and $X = \mathrm{SL}_2\mathbf{R}/\mathrm{SO}_2$ is the Poincaré disk. Then $\mathcal{T}$ is the Teichmüller space, which has dimension $6g - 6$.

The study of deformations of discontinuous groups for non-Riemannian homogeneous spaces and the failure of rigidity was initiated by Kobayashi [K93]; Kobayashi [K98] treats the case when G is semi-simple. A complete description of the parameter and deformation spaces was first given for $\Gamma = \mathbf{Z}^k$ acting on $X = \mathbf{R}^{k+1}$ via some nilpotent group of transformations G in [KN06] and these results were extended to the case where G is the Heisenberg group, H is any connected Lie subgroup and Γ is a subgroup acting properly discontinuously and freely on $X = G/H$, in [BKY].

In this paper, we give a concrete description of the space $R(\Gamma, G \times G; G)$, where G is the Heisenberg group, $\Gamma = G \cap \mathrm{GL}_3\mathbf{Z}$ is the discrete Heisenberg group and the direct product group $G \times G$ acts on the group manifold G from the left and right. Our main result is the following.

Theorem 1.2. *For the deformation space $\mathcal{T}(\Gamma, G \times G; G)$ of the discrete Heisenberg group acting properly discontinuously on the group manifold G from the left and right, we have the homeomorphism*

$$\mathcal{T}(\Gamma, G \times G; G) \cong \mathrm{GL}_2\mathbf{R} \times \mathbf{R}^\times \times \mathbf{R}^3.$$

2. Notation. Let G denote the Heisenberg group and $\Gamma = G \cap \mathrm{GL}_3\mathbf{Z}$ denote the discrete Heisenberg group. We will replace the matrix notation by defining

2000 Mathematics Subject Classification. Primary 32G08, 22E40, 22F30, 57S30.

doi: 10.3792/pjaa.89.55

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} := \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

We will fix a presentation $\Gamma = \langle \gamma_1, \gamma_2 \rangle$, where

$$\gamma_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \gamma_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \tag{1}$$

As a subgroup Γ always acts properly discontinuously and freely on G from the left and the quotient space $\Gamma \backslash G$ is a compact manifold. Similarly Γ always acts properly discontinuously from the right with compact quotient G/Γ.

To let Γ act both from the left and from the right, we rewrite G as the homogeneous space $G \times G/\Delta G$, where $\Delta: G \to G \times G$ is the diagonal embedding. Then Γ acts on $G \times G/\Delta G$ via homomorphisms $\Gamma \to G \times G$. We note here that $\mathrm{Hom}(\Gamma, G \times G) \cong (G \times G) \times (G \times G)$ as sets, because each generator γ_1, γ_2 can be assigned any element in $G \times G$, as any relations γ_1 and γ_2 satisfy as elements of G are also satisfied by any two arbitrary elements in $G \times G$. Via the topology on G, then, $\mathrm{Hom}(\Gamma, G \times G)$ can be regarded a topological space. In particular, for G being the Heisenberg group we have that $G \cong \mathbf{R}^3$, whence $\mathrm{Hom}(\Gamma, G \times G) \cong \mathbf{R}^{12}$.

Any homomorphism $\Gamma \to G \times G$ can be written as a pair of homomorphisms $\rho, \rho': \Gamma \to G$. Now write $\Gamma_{\rho,\rho'} = \{(\rho(\gamma), \rho'(\gamma)) \mid \gamma \in \Gamma\}$ for the image of the pair $(\rho, \rho'): \Gamma \to G \times G$. Then Γ acts on $G \times G/\Delta G$ via $\Gamma_{\rho,\rho'}$ and the action of Γ on G as subgroup (on the left) is recovered as the action of $\Gamma_{\mathrm{id},\mathbf{1}}$ on $G \times G/\Delta G$, where id is the inclusion and $\mathbf{1}$ is the trivial homomorphism. However, for general ρ, ρ' this action is not necessarily properly discontinuous.

Remark 2.1. Rewriting G as $G \times G/\Delta G$ for $G = \widetilde{\mathrm{SL}_2\mathbf{R}}$ allowed Goldman [G85] to construct non-standard Lorentz space forms. Goldman's conjecture concerning the existence of an open neighbourhood of the embedding $\mathrm{id} \times \mathbf{1}$, throughout which the group action remains properly discontinuous was resolved affirmatively for reductive Lie groups by Kobayashi [K98]. An analogous result holds if G is a simply connected Lie group and Γ is a cocompact discrete group by an unpublished result of T. Yoshino. Our results below show this feature explicitly for G being the Heisenberg group.

3. Property (CI) and proper actions. To check for proper discontinuity of the action of $\Gamma_{\rho,\rho'}$, we will use a criterion by Nasrin [N01] for 2-step nilpotent groups, which relates properness to the *property* (CI).

Definition 3.1 ([K92], Def. 6)**.** We say the triplet (L, H, G) has the property (CI) if $L \cap gHg^{-1}$ is compact for any $g \in G$.

(See [L95] for the relationship between the property (CI) and proper actions in the more general context of locally compact topological groups acting on locally compact topological spaces.)

Theorem 3.2 ([N01], Thm. 2.11)**.** *Let G be a simply connected 2-step nilpotent Lie group, and let H and L be connected subgroups. Then the following conditions are equivalent.*

(a) *L acts properly on G/H,*

(b) *the triplet (L, H, G) has the property* (CI)*,*

(c) *$L \cap gHg^{-1} = \{e\}$ for any $g \in G$.*

We will apply this theorem to the triple $(L_{\rho,\rho'}, \Delta G, G \times G)$, where G is again the Heisenberg group and $L_{\rho,\rho'}$ is the *extension* of $\Gamma_{\rho,\rho'}$ defined as follows.

Definition 3.3. Let Γ be a discrete subgroup in a Lie group G. A connected subgroup $L \subset G$ is said to be the *extension* of Γ if L contains Γ cocompactly.

The following lemma will allow us to use Thm. 3.2 to determine the conditions under which $\Gamma_{\rho,\rho'}$ acts properly discontinuously.

Lemma 3.4 ([K89])**.** *Let L be a Lie group acting continuously on a locally compact space X. Let $\Gamma \subset L$ be a discrete subgroup such that $\Gamma \backslash L$ is compact. Then the following conditions are equivalent.*

(a) *Γ acts properly discontinuously on X,*

(b) *L acts properly on X.*

4. Main results. To find the extension of $\Gamma_{\rho,\rho'}$, we use the (global) diffeomorphism $\exp: \mathfrak{g} \to G$, whose inverse we denote by log. Let $\rho, \rho': \Gamma \to G$ be any two homomorphisms. Then ρ and ρ' are determined by their values on the generators, which (in the notation of §2) we will set to be

$$\rho(\gamma_i) = \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix} \quad \text{and} \quad \rho'(\gamma_i) = \begin{bmatrix} a'_i \\ b'_i \\ c'_i \end{bmatrix}, \tag{2}$$

for $i = 1, 2$. Now, let $\rho_0: \mathfrak{g} \to \mathfrak{g}$ be a Lie algebra homomorphism defined on the generators by $\rho_0(\log \gamma_i) = \log \rho(\gamma_i)$, for $i = 1, 2$, and $\rho_0([\log \gamma_1, \log \gamma_2]) = \log \rho([\gamma_1, \gamma_2])$, and extended linearly; let

$\overline{\rho}: G \to G$ be defined by $\overline{\rho} = \exp \circ \rho_0 \circ \log$. Then $\overline{\rho}|_\Gamma = \rho$, so that $\overline{\rho}$ extends ρ in the sense that $\overline{\rho}$ is defined on all of G. If we write $\overline{\rho}'$ for the extension of ρ' to all of G, then $L_{\rho,\rho'} = \{(\overline{\rho}(g), \overline{\rho}'(g)) \mid g \in G\}$ is the extension of $\Gamma_{\rho,\rho'}$ in the sense of Def. 3.3.

Next, we will check condition (c) of Thm. 3.2 for $(L_{\rho,\rho'}, \Delta G, G \times G)$. We have that

$$
\begin{aligned}
& L_{\rho,\rho'} \cap (g_1, g_2)\Delta G(g_1, g_2)^{-1} = \{e\} \\
(3) \quad & \Leftrightarrow \overline{\rho}(g) = g_1^{-1} g_2 \overline{\rho}'(g)(g_1^{-1} g_2)^{-1} \text{ only if } g = e \\
& \Leftrightarrow \rho_0(\log g) = \mathrm{Ad}_{g_1^{-1} g_2} \rho_0'(\log g) \text{ only if } \log g = 0
\end{aligned}
$$

for all $(g_1, g_2) \in G \times G$. Now write

$$
g = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \log g = \begin{pmatrix} 0 & a & c - \frac{1}{2}ab \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}.
$$

Calculating the LHS and RHS of (3) explicitly, it follows that (3) is equivalent to

$$
\begin{pmatrix} a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \\ * & * & a_1 b_2 - a_2 b_1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a_1' & a_2' & 0 \\ b_1' & b_2' & 0 \\ * & * & a_1' b_2' - a_2' b_1' \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow a = b = c = 0.
$$

Writing

$$
(4) \quad A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \quad \text{and} \quad A' = \begin{pmatrix} a_1' & a_2' \\ b_1' & b_2' \end{pmatrix},
$$

we can rewrite condition (3) as

$$
\det \begin{pmatrix} A - A' & 0 \\ * & \det A - \det A' \end{pmatrix} \neq 0,
$$

and we obtain the following proposition.

Proposition 4.1. *The group $\Gamma_{\rho,\rho'}$ acts properly discontinuously and cocompactly on $G \times G/\Delta G$ if and only if the following two conditions hold.*
(a) $\det(A - A') \neq 0$, *and*
(b) $\det A - \det A' \neq 0$,
where A, A' are determined by ρ, ρ' via (2) *and* (4).

Proper discontinuity is contained in the above argument. For cocompactness we make use of the following lemma.

Lemma 4.2. *Let ρ be as in* (2) *and A be defined by* (4). *Then $\det A \neq 0 \Leftrightarrow \rho$ is injective.*

Proof. $\det A$ is precisely the $(1, 3)$ entry of the commutator $[\rho(\gamma_1), \rho(\gamma_2)]$ and $\det A \neq 0$ if and only if the image $\rho(\Gamma)$ is non-commutative. We show that $\rho(\Gamma)$ being non-commutative is equivalent to ρ being injective.

If ρ is injective, $\rho(\Gamma) \cong \Gamma$ is non-commutative. Conversely, write $N = \ker \rho$ and assume that $\rho(\Gamma)$ is non-commutative. We have the commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N \cap \mathbf{Z} & \longrightarrow & N & \longrightarrow & N/N \cap \mathbf{Z} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{Z} & \longrightarrow & \Gamma & \longrightarrow & \mathbf{Z}^2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathbf{Z}/N \cap \mathbf{Z} & \to & \Gamma/N & \longrightarrow & \Gamma/\mathbf{Z}N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

whose rows and columns are exact by the nine lemma. Turning our attention to the first column, the top left entry $N \cap \mathbf{Z}$ can be considered as a subgroup of $\mathbf{Z}$, and is thus equal to (*i*) 0, (*ii*) $\mathbf{Z}$, or (*iii*) $m\mathbf{Z}$, for some $m \geq 2$.

Case (*ii*). If $N \cap \mathbf{Z} = \mathbf{Z}$, N contains the commutator $\mathbf{Z} = [\Gamma, \Gamma]$, contradicting the fact that $\rho(\Gamma) \cong \Gamma/N$ was assumed non-commutative.

Case (*iii*). If $N \cap \mathbf{Z} = m\mathbf{Z}$, for $m \geq 2$, then $\mathbf{Z}/N \cap \mathbf{Z} = \mathbf{Z}_m$ in the bottom left entry. However, $\mathbf{Z}_m$ is finite and contains torsion elements and injects into Γ/N. By the first isomorphism theorem for groups, the induced map $\rho_*: \Gamma/N \to G$ is injective. But G is torsion-free, whence Γ/N is torsion-free also and we obtain a contradiction.

We conclude that $N \cap \mathbf{Z} = 0$ (case (*i*)).

Now, write $\pi: \Gamma \to \mathbf{Z}^2$ for the projection and $\pi^*: N \to N/N \cap \mathbf{Z}$ for the restriction of π to N. Let γ be any element in Γ and $n \in N$. Since N is normal, $\gamma n \gamma^{-1} \in N$. Then

$$
\pi^*(\gamma n \gamma^{-1}) = \pi^*(\gamma)\pi^*(n)\pi^*(\gamma^{-1}) = \pi^*(n),
$$

where the last equality follows from the fact that $\operatorname{im} \pi^*$ injects into $\mathbf{Z}^2$ and is therefore commutative. Since $N \cap \mathbf{Z} = 0$, π^* is an isomorphism and we conclude that $\gamma n = n \gamma$, i.e. N is contained in the centraliser $\mathbf{Z}$. Then $N \cap \mathbf{Z} = 0$ shows that N is trivial, whence ρ is injective. □

Proof of Prop. 4.1. Using Thm. 3.2, we have shown that $L_{\rho,\rho'}$ acts properly on $G \times G/\Delta G$ if and only if conditions (a) and (b) hold. Applying Lem. 3.4, $L_{\rho,\rho'}$ acts properly on $G \times G/\Delta G$ if and only if $\Gamma_{\rho,\rho'}$ acts properly discontinuously on $G \times G/\Delta G$.

By Lem. 4.2, condition (b) shows that at least one of ρ, ρ' must be injective, whence the cohomological dimension $\operatorname{cd}\Gamma_{\rho,\rho'} = 3$. It is a fact, based on a standard argument invoking Poincaré duality, that if a group Γ acts (faithfully) on a contractible manifold X and $\operatorname{cd}\Gamma = \dim X$, then $\Gamma\backslash X$ is compact (cf. [K89], Cor. 5.5). Since $G \times G/\Delta G \cong \mathbf{R}^3$ is indeed contractible and $\dim G \times G/\Delta G = \operatorname{cd}\Gamma_{\rho,\rho'} = 3$, the double quotient $\Gamma_{\rho,\rho'}\backslash G \times G/\Delta G$ is compact. □

Prop. 4.1 can be turned into a method for determining pairs of homomorphisms for which $\Gamma_{\rho,\rho'}$ acts properly discontinuously and cocompactly on G from the left and right as follows.

Let

$$S = \begin{pmatrix} s_0 & s_1 \\ s_2 & s_3 \end{pmatrix} \in \mathrm{GL}_2\mathbf{R}$$

and $(S, t_0, t_1, t_2, t_3, c_1, c_2, c_1', c_2') \in \mathrm{GL}_2\mathbf{R} \times \mathbf{R}^\times \times \mathbf{R}^7$. Define a map

$$\alpha\colon \mathrm{GL}_2\mathbf{R} \times \mathbf{R}^\times \times \mathbf{R}^7 \to R(\Gamma, G \times G; G) \tag{5}$$
$$(S, t_0, t_1, t_2, t_3, c_1, c_2, c_1', c_2') \mapsto \phi,$$

where $\phi = (\rho, \rho')$ is defined by

$$\rho(\gamma_1) = \begin{bmatrix} \frac{1}{2}(s_0(t_0+t_3) + s_0 + s_1 t_2) \\ \frac{1}{2}(s_2(t_0+t_3) + s_2 + s_3 t_2) \\ c_1 \end{bmatrix},$$

$$\rho(\gamma_2) = \begin{bmatrix} \frac{1}{2}(s_1(t_0-t_3) + s_1 + s_0 t_1) \\ \frac{1}{2}(s_3(t_0-t_3) + s_3 + s_2 t_1) \\ c_2 \end{bmatrix},$$

$$\rho'(\gamma_1) = \begin{bmatrix} \frac{1}{2}(s_0(t_0+t_3) - s_0 + s_1 t_2) \\ \frac{1}{2}(s_2(t_0+t_3) - s_2 + s_3 t_2) \\ c_1' \end{bmatrix},$$

$$\rho'(\gamma_2) = \begin{bmatrix} \frac{1}{2}(s_1(t_0-t_3) - s_1 + s_0 t_1) \\ \frac{1}{2}(s_3(t_0-t_3) - s_3 + s_2 t_1) \\ c_2' \end{bmatrix}.$$

Determining A, A' via (4), one checks that $A - A' = S$ and $\det A - \det A' = t_0 \cdot \det S \neq 0$, as $t_0 \in \mathbf{R}^\times$. Thus, conditions (a) and (b) from Prop. 4.1 are satisfied and $\Gamma_{\rho,\rho'}$ acts properly discontinuously and cocompactly on G from the left and right. Moreover, we have the following theorem.

Theorem 4.3. *The map α (see (5)) induces a homeomorphism from $\mathrm{GL}_2\mathbf{R} \times \mathbf{R}^\times \times \mathbf{R}^7$ onto the parameter space $R(\Gamma, G \times G; G)$ of deformations of Γ acting properly discontinuously on the group manifold G from the left and right. Furthermore, the deformation space $\mathcal{T}(\Gamma, G \times G; G)$ is homeomorphic to $\mathrm{GL}_2\mathbf{R} \times \mathbf{R}^\times \times \mathbf{R}^3$.*

Proof. The idea of the proof and the origin of the map α is the following.

The space of pairs of matrices satisfying (a) and (b) of Prop. 4.1 can be determined as follows. Suppose A, A' satisfy (a) and (b). Consider the map

$$\omega\colon (A, A') \mapsto (U, V) = (A - A', (A - A')^{-1}(A + A')),$$

which is well-defined, since $U = A - A'$ is invertible. We can find an inverse mapping

$$\alpha_0\colon (U, V) \mapsto (\tfrac{1}{2}(UV + U), \tfrac{1}{2}(UV - U))$$

and one checks that $\alpha_0 \circ \omega = \mathrm{id}$ and $\omega \circ \alpha_0 = \mathrm{id}$.

For U and V, condition (a) is equivalent to the condition that $U \in \mathrm{GL}_2\mathbf{R}$; condition (b) translates into the condition

$$\begin{aligned} & \det\tfrac{1}{2}(UV + U) \neq \det\tfrac{1}{2}(UV - U) \\ \Leftrightarrow\quad & \det(V + I) \neq \det(V - I), \\ \Leftrightarrow\quad & \det V + \operatorname{tr} V \neq \det V - \operatorname{tr} V \\ \Leftrightarrow\quad & \operatorname{tr} V \neq 0, \end{aligned}$$

where I denotes the 2×2 identity matrix. Then, writing $M = \{V \in M_2(\mathbf{R}) \mid \operatorname{tr} V \neq 0\} \cong \mathbf{R}^\times \times \mathbf{R}^3$, the map

$$\alpha_0\colon \mathrm{GL}_2\mathbf{R} \times M \to \{(A, A') \mid A, A' \text{ satify (a) \& (b)}\}$$

is a homeomorphism. Writing id for the identity on $\mathbf{R}^4 = \{(c_1, c_2, c_1', c_2') | c_1, c_2, c_1', c_2' \in \mathbf{R}\}$, $\alpha_0 \times \mathrm{id} = \alpha$ is the homeomorphism

$$\alpha\colon \mathrm{GL}_2\mathbf{R} \times \mathbf{R}^\times \times \mathbf{R}^7 \to R(\Gamma, G \times G; G)$$

up to the identification $M \cong \mathbf{R}^\times \times \mathbf{R}^3$.

The conjugation action of $G \times G$ on $\Gamma_{\rho,\rho'}$ leaves the superdiagonal entries of each factor unchanged and is transitive on the $(1,3)$ entries, so that $\mathcal{T}(\Gamma, G \times G; G) = R(\Gamma, G \times G; G)/(G \times G)$ is homeomorphic to

$$\mathrm{GL}_2\mathbf{R} \times \mathbf{R}^\times \times \mathbf{R}^3.$$

□

5. Geometric interpretation of main result. Geometrically speaking, we have the central extensions

$$0 \to \mathbf{R} \to G \to \mathbf{R}^2 \to 0$$
$$0 \to \mathbf{Z} \to \Gamma \to \mathbf{Z}^2 \to 0$$

and by quotienting $\Gamma\backslash G$ can be viewed as a circle bundle over the torus. The two conditions of Prop. 4.1 can then be interpreted as follows. The matrix $A - A'$ determines a Riemannian structure on this torus and $\det A - \det A'$ determines the structure on (i.e. length of) the circle. In particular, the number of connected components (which equals four) of the deformation space $\mathcal{T}(\Gamma, G \times G; G)$ corresponds to the number of possible combinations of orientations on the torus and the circle.

Example 5.1. Let

$$\rho(\gamma_1) = \begin{bmatrix} 2 \\ c \\ 0 \end{bmatrix}, \qquad \rho(\gamma_2) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

and

$$\rho'(\gamma_1) = \begin{bmatrix} 1 \\ c \\ 0 \end{bmatrix}, \qquad \rho'(\gamma_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Letting c vary from 0 to 1, we obtain a family of groups $\Gamma_{\rho,\rho'}$ (which lies in the component of both base space and fibre orientations being positive), which by Prop. 4.1 act cocompactly and properly discontinuously on $G \times G/\Delta G$, where the length of the fibre varies from 3 to 2 and the structure on the torus remains unchanged and is given by the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

Similarly, it is possible to find families of groups, which only change the structure on the base space, leaving the length of the fibre unchanged; or families, for which both the structure on the base space and the length of the fibre are fixed, but the connection form is deformed.

Remark 5.2. General examples, like the one above, stand in contrast to the case when G is semisimple of real rank 1—e.g. $G = \mathrm{SL}_2\mathbf{R}$, $\mathrm{SO}(n,1)$, $\mathrm{SU}(n,1)$, $\mathrm{Sp}(n,1)$—for which any properly discontinuous group for $G \times G/\Delta G$ is a graph up to a finite-index subgroup ([K93], Thm. 2 and Rmk. 1).

Acknowledgements. The author would like to thank Prof. Taro Yoshino for detailed comments on an earlier version of this paper and Prof. Toshiyuki Kobayashi for his comments, guidance and invaluable advice.

References

[BKY] A. Baklouti, I. Kédim and T. Yoshino, On the deformation space of Clifford–Klein forms of Heisenberg groups, Int. Math. Res. Not. IMRN **2008**, no. 16, Art. ID rnn066, 35 pp.

[G85] W. M. Goldman, Nonstandard Lorentz space forms, J. Differential Geom. **21** (1985), no. 2, 301–308.

[K89] T. Kobayashi, Proper action on a homogeneous space of reductive type, Math. Ann. **285** (1989), no. 2, 249–263.

[K92] T. Kobayashi, Discontinuous groups acting on homogeneous spaces of reductive type, in *Representation theory of Lie groups and Lie algebras* (*Fuji-Kawaguchiko, 1990*), 59–75, World Sci. Publ., River Edge, NJ, 1992.

[K93] T. Kobayashi, On discontinuous groups acting on homogeneous spaces with noncompact isotropy subgroups, J. Geom. Phys. **12** (1993), no. 2, 133–144.

[K98] T. Kobayashi, Deformation of compact Clifford–Klein forms of indefinite-Riemannian homogeneous manifolds, Math. Ann. **310** (1998), no. 3, 395–409.

[K01] T. Kobayashi, Discontinuous groups for non-Riemannian homogeneous spaces, in *Mathematics unlimited—2001 and beyond*, 723–747, Springer, Berlin, 2001.

[KN06] T. Kobayashi and S. Nasrin, Deformation of properly discontinuous actions of $\mathbf{Z}^k$ on $\mathbf{R}^{k+1}$, Internat. J. Math. **17** (2006), no. 10, 1175–1193.

[L95] R. L. Lipsman, Proper actions and a compactness condition, J. Lie Theory **5** (1995), no. 1, 25–39.

[N01] S. Nasrin, Criterion of proper actions for 2-step nilpotent Lie groups, Tokyo J. Math. **24** (2001), no. 2, 535–543.

[W64] A. Weil, Remarks on the cohomology of groups, Ann. of Math. (2) **80** (1964), 149–157.