Some restricted sum formulas for double zeta values

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Abstract: We give some restricted sum formulas for double zeta values whose arguments satisfy certain congruence conditions modulo 2 or 6, and also give an application to identities showed by Ramanujan for sums of products of Bernoulli numbers with a gap of 6.

Key words: Multiple zeta value; double zeta value; sum formula; Bernoulli number; Ramanujan's identity.

1. Introduction. The double zeta values are defined by

$$\zeta(l_1, l_2) := \sum_{m_1 > m_2 > 0} \frac{1}{m_1^{l_1} m_2^{l_2}} \tag{1.1}$$

for integers $l_1 \geq 2, l_2 \geq 1$. These values were studied in detail in [3], and interesting facts such as linear relations and connections with modular forms (especially period polynomials) were discovered.

Historically, Euler [2] first studied these values, and showed the sum formula

$$\sum_{\substack{l_1 \geq 2, l_2 \geq 1 \\ (l_1 + l_2 = l)}} \zeta(l_1, l_2) = \zeta(l). \tag{1.2}$$

When the weight $l = l_1 + l_2$ is even, Gangl, Kaneko and Zagier [3] gave restricted analogues of the sum formula, more precisely, proved the following formulas for double zeta values with even and odd arguments.

$$\sum_{l_1, l_2 \equiv 0(2)}{}' \zeta(l_1, l_2) = \frac{3}{4}\zeta(l), \tag{1.3}$$
$$\sum_{l_1, l_2 \equiv 1(2)}{}' \zeta(l_1, l_2) = \frac{1}{4}\zeta(l),$$

where $\sum'_{c(l_1,l_2)}$ means running over the integers l_1, l_2 satisfying $l_1 \geq 2, l_2 \geq 1, l = l_1 + l_2$ and the condition $c(l_1, l_2)$. Nakamura [4] pointed out that the first formula of (1.3) yields the identity showed by Euler for sums of products of Bernoulli numbers

$$\sum_{\substack{j=0 \\ (j \equiv 0(2))}}^{l} \binom{l}{j} B_j B_{l-j} = -(l-1)B_l \quad (l \geq 4), \tag{1.4}$$

and vice versa, where the Bernoulli numbers B_m are defined by $X/(e^X - 1) = \sum_{m=0}^{\infty}(B_m/m!)X^m$.

In this paper, we give some new restricted sum formulas for double zeta values of any weight l whose first arguments l_1 satisfy certain congruence conditions modulo 2 or 6, and prove that an obtained restricted sum formula yields identities showed by Ramanujan for sums of products of Bernoulli numbers with a gap of 6, and vice versa.

The restricted sum formulas are as follows, which are divided into three classes according to the value of the weight modulo 3.

Theorem 1.1. *Let l be an integer such that $l \geq 3$, and let the empty sum mean 0.*

(i) *If $l \equiv 0(3)$,*

$$\left(\sum_{l_1 \equiv 3(6)}{}' - \sum_{l_1 \equiv 4(6)}{}' - \sum_{l_1 \equiv 5(6)}{}'\right)\zeta(l_1, l_2) = \frac{1}{3}\sum_{l_1 \equiv 1(2)}{}' \zeta(l_1, l_2). \tag{1.5}$$

(ii) *If $l \equiv 1(3)$,*

$$\left(\sum_{l_1 \equiv 3(6)}{}' + \sum_{l_1 \equiv 4(6)}{}' - \sum_{l_1 \equiv 5(6)}{}'\right)\zeta(l_1, l_2) = \frac{1}{3}\sum_{l_1 \equiv 0(2)}{}' \zeta(l_1, l_2). \tag{1.6}$$

(iii) *If $l \equiv 2(3)$,*

$$\sum_{l_1 \equiv 4(6)}{}' \zeta(l_1, l_2) = \frac{1}{6}\zeta(l) - \frac{1}{3}\sum_{l_1 \equiv 1(2)}{}' \zeta(l_1, l_2). \tag{1.7}$$

We restate the restricted sum formulas in the case where l is even as a corollary, since the restated formulas include (1.10) which yields the identities

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showed by Ramanujan, and the other formulas seem interesting in themselves. Restating is easily carried out by (1.3) and the Chinese remainder theorem.

Corollary 1.2. *Let l be an even integer such that $l \geq 4$.*

(i) *If $l \equiv 0(6)$,*

$$\left(\sum_{l_1,l_2\equiv 3(6)}{}' - \sum_{\substack{l_1\equiv 4(6)\\ l_2\equiv 2(6)}}{}' - \sum_{\substack{l_1\equiv 5(6)\\ l_2\equiv 1(6)}}{}'\right)\zeta(l_1,l_2) = \frac{1}{12}\zeta(l). \tag{1.8}$$

(ii) *If $l \equiv 4(6)$,*

$$\left(\sum_{\substack{l_1\equiv 3(6)\\ l_2\equiv 1(6)}}{}' + \sum_{\substack{l_1\equiv 4(6)\\ l_2\equiv 0(6)}}{}' - \sum_{l_1,l_2\equiv 5(6)}{}'\right)\zeta(l_1,l_2) = \frac{1}{4}\zeta(l). \tag{1.9}$$

(iii) *If $l \equiv 2(6)$,*

$$\sum_{l_1,l_2\equiv 4(6)}{}' \zeta(l_1,l_2) = \frac{1}{12}\zeta(l). \tag{1.10}$$

Ramanujan [6, (13)] (see also [7, (13)]) showed the following identities for sums of products of Bernoulli numbers with a gap of 6,

$$\sum_{\substack{j=0\\ (j\equiv m(6))}}^{l} \binom{l}{j} B_j B_{l-j} = -\frac{l-1}{3}B_l \quad (m=0,2,4) \tag{1.11}$$

where $l \equiv 2(6)$ and $l \geq 8$. To be precise, he proved only (1.11) with $m = 0$ by using identities of trigonometric functions, but it is easily seen that the three identities in (1.11) are equivalent; Identities (1.11) with $m = 0$ and 2 are derived from the index change $j \to l - j$ each other, and the two identities yield (1.11) with $m = 4$ and vice versa because of (1.4). Note that Ramanujan considered Bernoulli numbers to be not B_m but $|B_m|$ for positive even integers m in [6], and that there is a minor misprint in [7, (13)], that is, the right hand side of [7, (13)] should be multiplied by B_{6n+2}. Though identities of Bernoulli numbers have been studied for a very long time and rediscovered many times, (1.11) seems truly due to Ramanujan by Wagstaff's comment in [7, p. 54] (see also [1, Chapter 5] for Ramanujan's works about Bernoulli numbers).

We have the following corollary, which gives a new proof of (1.11) via double zeta values.

Corollary 1.3. (1.10) *yields* (1.11) *and vice versa.*

In the next and final section, we prove Theorem 1.1 and Corollary 1.3.

2. Proofs. In order to prove Theorem 1.1, we refer to the proof of (1.3) in [3], that is, we will use linear combinations of special values of the polynomials which are defined by

$$\mathfrak{D}_l(x,y) := \sum{}' x^{l_1-1}y^{l_2-1}\zeta(l_1,l_2) \tag{2.1}$$

for integers $l \geq 3$. In fact, the formulas of (1.3) are obtained by

$$\sum_{l_1\equiv 0(2)}{}' \zeta(l_1,l_2) = \frac{\mathfrak{D}_l(1,1) - \mathfrak{D}_l(-1,1)}{2},$$

$$\sum_{l_1\equiv 1(2)}{}' \zeta(l_1,l_2) = \frac{\mathfrak{D}_l(1,1) + \mathfrak{D}_l(-1,1)}{2},$$

since $\mathfrak{D}_l(1,1) = \zeta(l)$ due to (1.2) and $\mathfrak{D}_l(-1,1) = -\zeta(l)/2$ if l is even (see [3, §2]).

For a real number x, let $[x]$ and $\{x\}$ respectively denote the integer and fractional parts of x such that $x = [x] + \{x\}$, $[x] \in \mathbf{Z}$ and $0 \leq \{x\} < 1$.

The following proposition is necessary for the proof of Theorem 1.1.

Proposition 2.1. *For any integer $l \geq 3$, we have*

$$\begin{aligned}&\left(\sum_{\substack{l_1\equiv 2l(3)\\ l_1\equiv 1(2)}}{}' - \sum_{\substack{l_1\equiv 2l(3)\\ l_1\equiv 0(2)}}{}' - \sum_{l_1\equiv l-1(3)}{}' - 2\sum_{l_1\equiv 4(6)}{}'\right)\zeta(l_1,l_2)\\ &\quad = -\left\{\frac{l+1}{3}\right\}\zeta(l) + \frac{2}{3}\mathfrak{D}_l(-1,1).\end{aligned} \tag{2.2}$$

Proof. We see from [3, (26)] that

$$\begin{aligned}&\mathfrak{D}_l(x+y,y) + \mathfrak{D}_l(y+x,x)\\ &\quad = \mathfrak{D}_l(x,y) + \mathfrak{D}_l(y,x) + \frac{x^{l-1}-y^{l-1}}{x-y}\zeta(l).\end{aligned} \tag{2.3}$$

Let ω denote $\exp(2\pi i/3)$. By summing up (2.3) with $(x,y) = (1,1), (\omega,1), (\omega^2,1)$ and by Lemma 2.2 below, we get

$$\begin{aligned}&\left(\sum_{l_1\equiv 1(3)}{}' + \sum_{l_1\equiv 2l(3)}{}'\right)(-1)^{l_1-1}\zeta(l_1,l_2)\\ &\quad + \frac{l+1}{3}\zeta(l) - \frac{2}{3}\mathfrak{D}_l(-1,1)\\ &= \left(\sum_{l_1\equiv 1(3)}{}' + \sum_{l_1\equiv l-1(3)}{}'\right)\zeta(l_1,l_2) + \left[\frac{l+1}{3}\right]\zeta(l).\end{aligned} \tag{2.4}$$

A calculation shows that

$$\left(\sum_{l_1\equiv 1(3)}{}' + \sum_{l_1\equiv 2l(3)}{}'\right)(-1)^{l_1-1}\zeta(l_1,l_2) \tag{2.5}$$

$$- \left(\sum_{l_1 \equiv 1(3)}{}' + \sum_{l_1 \equiv l-1(3)}{}' \right) \zeta(l_1, l_2)$$

$$= \left(\sum_{\substack{l_1 \equiv 1(3) \\ l_1 \equiv 1(2)}}{}' - \sum_{\substack{l_1 \equiv 1(3) \\ l_1 \equiv 0(2)}}{}' + \sum_{\substack{l_1 \equiv 2l(3) \\ l_1 \equiv 1(2)}}{}' \right.$$

$$\left. - \sum_{\substack{l_1 \equiv 2l(3) \\ l_1 \equiv 0(2)}}{}' - \sum_{l_1 \equiv 1(3)}{}' - \sum_{l_1 \equiv l-1(3)}{}' \right) \zeta(l_1, l_2)$$

$$= \left(\sum_{\substack{l_1 \equiv 2l(3) \\ l_1 \equiv 1(2)}}{}' - \sum_{\substack{l_1 \equiv 2l(3) \\ l_1 \equiv 0(2)}}{}' - \sum_{l_1 \equiv l-1(3)}{}' - 2 \sum_{\substack{l_1 \equiv 1(3) \\ l_1 \equiv 0(2)}}{}' \right) \zeta(l_1, l_2).$$

Since $l_1 \equiv 1(3)$ and $l_1 \equiv 0(2)$ if and only if $l_1 \equiv 4(6)$, (2.4) and (2.5) prove (2.2). □

Lemma 2.2. *Let* $\sum_\omega$ *mean* $\sum_{x \in \{1, \omega, \omega^2\}}$. *For any integer* $l \geq 3$, *we have*

$$\sum_\omega \mathfrak{D}_l(x+1, 1) = 3 \sum_{l_1 \equiv 1(3)}{}' (-1)^{l_1 - 1} \zeta(l_1, l_2) + \frac{l+1}{2} \zeta(l) - \mathfrak{D}_l(-1, 1),$$

$$\sum_\omega \mathfrak{D}_l(x+1, x) = 3 \sum_{l_1 \equiv 2l(3)}{}' (-1)^{l_1 - 1} \zeta(l_1, l_2) + \frac{l+1}{2} \zeta(l) - \mathfrak{D}_l(-1, 1),$$

$$\sum_\omega \mathfrak{D}_l(x, 1) = 3 \sum_{l_1 \equiv 1(3)}{}' \zeta(l_1, l_2),$$

$$\sum_\omega \mathfrak{D}_l(1, x) = 3 \sum_{l_1 \equiv l-1(3)}{}' \zeta(l_1, l_2),$$

$$\left(\sum_\omega \frac{x^{l-1} - 1}{x - 1} \right) \zeta(l) = 3 \left[\frac{l+1}{3} \right] \zeta(l).$$

Proof. Let k be an integer. Because ω is the 3-th root of unity, $1 + \omega^k + \omega^{2k}$ is equal to 3 if $k \equiv 0(3)$ and 0 otherwise, in particular, $1 + \omega + \omega^2 = 0$. By using the weighted sum formula $\sum' 2^{l_1 - 1} \zeta(l_1, l_2) = (l+1)\zeta(l)/2$ given in [5], it follows from (2.1) that

$$\sum_\omega \mathfrak{D}_l(x+1, 1)$$
$$= \sum{}' (2^{l_1 - 1} + (-\omega)^{l_1 - 1} + (-\omega^2)^{l_1 - 1}) \zeta(l_1, l_2)$$
$$= \sum{}' (2^{l_1 - 1} - (-1)^{l_1 - 1} + (-1)^{l_1 - 1}(1 + \omega^{l_1 - 1} + \omega^{2(l_1 - 1)})) \zeta(l_1, l_2)$$
$$= \frac{l+1}{2} \zeta(l) - \mathfrak{D}_l(-1, 1) + 3 \sum_{l_1 \equiv 1(3)}{}' (-1)^{l_1 - 1} \zeta(l_1, l_2),$$

which verifies the first equation in the lemma. The other equations can be proved in the same way, and we omit the proofs. □

We prove Theorem 1.1.

Proof of Theorem 1.1. We will prove only (1.5) since we can do (1.6) and (1.7) similarly. Assume that $l \equiv 0(3)$. Then the left hand side of (2.2) is equal to

$$\left(\sum_{l_1 \equiv 3(6)}{}' - \sum_{l_1 \equiv 0(6)}{}' - \sum_{l_1 \equiv 2(6)}{}' - \sum_{l_1 \equiv 5(6)}{}' - 2 \sum_{l_1 \equiv 4(6)}{}' \right) \zeta(l_1, l_2)$$

$$= \left(\sum_{l_1 \equiv 3(6)}{}' - \sum_{l_1 \equiv 4(6)}{}' - \sum_{l_1 \equiv 5(6)}{}' - \sum_{l_1 \equiv 0(2)}{}' \right) \zeta(l_1, l_2)$$

$$= \left(\sum_{l_1 \equiv 3(6)}{}' - \sum_{l_1 \equiv 4(6)}{}' - \sum_{l_1 \equiv 5(6)}{}' \right) \zeta(l_1, l_2) - \frac{\mathfrak{D}_l(1,1) - \mathfrak{D}_l(-1,1)}{2},$$

and the right hand side is equal to

$$-\frac{1}{3} \mathfrak{D}_l(1,1) + \frac{2}{3} \mathfrak{D}_l(-1,1).$$

We thus obtain

$$\left(\sum_{l_1 \equiv 3(6)}{}' - \sum_{l_1 \equiv 4(6)}{}' - \sum_{l_1 \equiv 5(6)}{}' \right) \zeta(l_1, l_2) = \frac{\mathfrak{D}_l(1,1) + \mathfrak{D}_l(-1,1)}{6},$$

which proves (1.5). □

Finally we prove Corollary 1.3.

Proof of Corollary 1.3. We will derive (1.11) from (1.10). Since the identities in (1.11) yield each other by virtue of (1.4), we may only prove (1.11) with $m = 4$. From the harmonic relations $\zeta(l_1)\zeta(l_2) = \zeta(l_1, l_2) + \zeta(l_2, l_1) + \zeta(l)$, we see that

$$\sum_{l_1, l_2 \equiv 4(6)}{}' \zeta(l_1, l_2) = \frac{1}{2} \sum_{l_1, l_2 \equiv 4(6)}{}' (\zeta(l_1, l_2) + \zeta(l_2, l_1))$$

$$= \frac{1}{2} \sum_{l_1, l_2 \equiv 4(6)}{}' (\zeta(l_1)\zeta(l_2) - \zeta(l))$$

$$= \frac{1}{2} \left(\sum_{\substack{j=0 \\ (j \equiv 4(6))}}^{l} \zeta(j)\zeta(l-j) - \frac{l-2}{6}\zeta(l) \right).$$

This with (1.10) gives

$$\sum_{\substack{j=0 \\ (j \equiv 4(6))}}^{l} \zeta(j)\zeta(l-j) = \frac{1}{6}\zeta(l) + \frac{l-2}{6}\zeta(l)$$

$$= \frac{l-1}{6}\zeta(l).$$

By Euler's formula $\zeta(m) = -\frac{(2\pi i)^m}{2m!} B_m$ for any positive even integer m, we obtain (1.11) with $m = 4$.

The converse follows by the reversing the above statements. □

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