Alexander polynomials of certain dual of smooth quartics

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Abstract: In this paper, we compute Alexander polynomials of the dual curves of certain smooth quartic curves. From our previous paper, all of these dual curves are $(2,3)$ torus curves of degree 12. As a consequence, from these curves, we find a new Zariski pair $12E_6 + 16A_1$, with different Alexander polynomials.

Key words: Zariski pair; Alexander polynomial; dual curve; torus curve.

1. Introduction. Given a homogeneous polynomial F in $\mathbf{C}[x,y,z]$, in [4], a *quasi-toric decomposition of* F is a collection of homogeneous polynomials $f,g,h \in \mathbf{C}[x,y,z]$ such that $f^p + g^q + h^{pq}F = 0$, for two co-prime positive integers $p,q > 1$.

In particular, a curve is called of *torus type* (p,q) if it admits a quasi-toric decomposition where $h = 1$. For short, we call a (p,q) torus curve. Such a curve was first studied by Zariski in [15] where a sextic of torus type $(2,3)$ appears as the dual of a smooth cubic curve. Also in that paper, the first example of a Zariski pair was given. Briefly, a *Zariski pair* is a pair of curves with the same local type of singularities but have different types of topology of the complement in $\mathbf{CP}^2$. This pair consists of two sextics with 6 cusps A_2, one of these two curves has the property that 6 cusps are lying on a conic whereas another curve has no such the property. Later it is known that the first curve is a $(2,3)$ torus curve, the latter is not.

For the last twenty years after the paper of Artal [1] was published, the study of Zariski pairs is an active area. The main tools are the fundamental group of the complement $\pi_1(\mathbf{P}^2 \setminus C)$ and the Alexander polynomial. Many Zariski pairs of sextics were given in [10–12], most of them appear as a $(2,3)$ torus curve and a non $(2,3)$ torus curve.

The next candidates of $(2,3)$ torus curves are curves of degree 12. In this paper we give a new example of Zariski pair, both of these curves are $(2,3)$ torus curves. This is, in our knowledge, the first example of a Zariski pair where both are (p,q) torus curves.

Furthermore the $(2,3)$ torus curves of degree 12 in this paper come as dual curves of smooth quartics (see our result [13]). This is an analog case of Zariski's sextic with $9A_2$ which is the dual of a smooth cubic. The Alexander polynomial of this sextics with $9A_2$ is $(t^2 - t + 1)^3$. This is the highest degree polynomial among the Alexander polynomials of sextics of torus type (see [8]). To compute Alexander polynomials of the dual curves of the smooth quartics is the motivation for our interest.

Our main results in this paper are Propositions 4.1 and 4.2 found in the last section.

2. Dual of smooth quartics. In this section we first recall the definition of the dual curve. Later we recall some results about the dual curves of smooth quartics and their $(2,3)$ toric decomposition found in our previous paper [13].

Definition 2.1. Let $C \subset \mathbf{P}^2$ be a curve. Let us consider the set of lines in $\mathbf{P}^2$ as another projective space, $(\mathbf{P}^2)^*$, where a point (a,b,c) in $(\mathbf{P}^2)^*$ corresponds to the line $aX + bY + cZ = 0$ in $\mathbf{P}^2$. Then the closure of the set $\{T_P C \in (P^2)^* | P \in C \setminus Sing(C)\}$ is called the *dual curve* of C, which we denote by C^*.

It is well-known that the dual of a (generic) smooth cubic is a sextic with exactly $9A_2$'s and the dual of a generic smooth quartic is a curve of degree 12 with $28A_1$'s and $24A_2$'s. In general, if a smooth quartic curve has n hyperflexes, then the singularities of the dual curve are $(28-n)A_1 + (24-2n)A_2 + nE_6$, ($0 \le n \le 12$, $n \ne 10$ and $n \ne 11$).

As we mentioned in the introduction, the dual of a smooth cubic has a $(2,3)$ toric decom-

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position. In [14], Tokunaga showed that there are at least 12 different toric decompositions. Moreover, we proved in [13] that there are exactly 12 decompositions.

In [13] we proved a similar result for smooth quartics, the dual of any smooth quartic is a $(2,3)$ torus curves of degree 12. About the singularities on the dual curves: $(24-2n)A_2 + nE_6$ locate at the same positions as the intersection points of $V(g_4)$ and $V(g_6)$, where $g_4^3 + g_6^2$ is the toric decomposition of the dual curve; $(28-n)A_1$ singularities stay outside of the intersection of $V(g_4)$ and $V(g_6)$.

In this paper, we study three interesting cases: The Klein curve K_4 (no hyperflex, $n = 0$), the Fermat curve F_4 and another partner which is named by G_4 (maximum number of hyperflexes, $n = 12$ in both cases).

For the Klein curve K_4, which is defined by $x^3y + y^3z + z^3x = 0$, the dual curve K_4^* has a toric decomposition $27g_6^2 - 4g_4^3$ (we can rewrite it to the standard form), where

$$g_4 = u^3v + v^3w + w^3u$$
$$g_6 = uv^5 + vw^5 + wu^5 - 5u^2v^2w^2,$$

are both irreducible polynomials. The dual curve K_4^* has $24A_2 + 28A_1$ singularities.

The Fermat quartic curve F_4: Its defining polynomial is $x^4 + y^4 + z^4$. The toric decomposition of F_4^* is given by

$$f^* = -27(u^2v^2w^2)^2 + (u^4 + v^4 + w^4)^3.$$

The last curve G_4 which also shares the same number of hyperflexes as the Fermat curve is defined by

$$x^4 + y^4 + z^4 + 3(x^2y^2 + y^2z^2 + z^2x^2) = 0.$$

The toric decomposition of G_4^* is given by $g_4^3 - 27g_6^2$, where

$$g_6 = (3u^2 + v^2 + w^2)(u^2 + 3v^2 + w^2)(u^2 + v^2 + 3w^2)$$
$$g_4 = 7(u^4 + v^4 + w^4) + 18(u^2v^2 + v^2w^2 + w^2u^2).$$

In the last two curves, their dual curves F_4^* and G_4^* have the same $12E_6 + 16A_1$ singularities.

3. Alexander polynomials of curves. In this section, we recall some facts about the Alexander polynomial of a plane curve. As we mentioned in the introduction, the fundamental group of the complement and the Alexander polynomial are the main tools to show that a pair of curves is a Zariski pair. Normally, the fundamental groups are not easy to compute, the Alexander polynomial which is introduced by Libgober [5] in the 80s is a weaker invariant, but it is easier to compute. Recently, Libgober wrote a nice survey on this invariant in [6]. Together with another survey by Oka [9], one can learn several ways to compute this polynomial. Here, we briefly recall some facts for our purpose.

Let consider an irreducible curve C of degree d. Denote $Sing(C)$, or $Sing$ for short, the set of singularities of C.

Lemma 3.1 (Libgober, in [5]). *The Alexander polynomial $\Delta(t)$ divides the product of the local Alexander polynomials $\prod_{P\in Sing(C)} \Delta_P(t)$.*

For example, the Alexander polynomial of an irreducible curve with at most cusps and nodes is equal to $(t^2 - t + 1)^s$ for some non-negative integer s. The reason is that the local Alexander polynomial of a node is 1 and of a cusp is $t^2 - t + 1$.

Thanks to the works of Artal, Esnault and Loeser-Vaquié (cf. [1,3,7]), the Alexander polynomial can be computed explicitly as follows:

Lemma 3.2. *The Alexander polynomial is written as*

$$\Delta(t) = \prod_{k=1}^{d-1} \Delta_k(t)^{\ell_k}, \quad k = 1, \dots, d-1,$$

where $\Delta_k(t) = (t - \exp(\frac{2k\pi i}{d}))(t - \exp(\frac{-2k\pi i}{d}))$ *and* $\ell_k = \dim H^1(\mathbf{P}^2, \mathcal{L}^{(k)})$.

We refer to [1,3,7] for the definition of the sheaf $\mathcal{L}^{(k)}$. By [1], the integer ℓ_k is equal to the dimension of the cokernel of the homomorphism

$$\sigma_k : H^0(\mathbf{P}^2, \mathcal{O}(k-3)) \to \oplus_{P\in Sing}\mathcal{O}_p/\mathcal{J}_{P,k,d}.$$

Remark 3.3. For detailed explanations and examples about the ideal $\mathcal{J}_{P,k,d}$ and the cokernel of σ_k we refer to [1,8,9].

In this paper, except the trivial case of A_1, the ideals $\mathcal{J}_{P,k,d}$ where $d = 12$ are computed as below.

- A_2 $(u^2 + v^3 = 0)$ case: $\langle u, v\rangle$ (for $k = 10$ or 11) and trivial otherwise.
- E_6 $(u^3 + v^4 = 0)$ case: $\langle u, v\rangle$ (for $k = 7, 8, 9$), $\langle u, v^2\rangle$ (for $k = 10$), $\langle u^2, uv, v^2\rangle$ (for $k = 11$) and trivial otherwise.

The following result is the second part of Theorem 4.7 in [2].

Theorem 3.4 (Cogolludo-Libgober [2]). *For any irreducible plane curve $C = \{F = 0\}$ whose only singularities are nodes and cusps, the set of quasi-*

toric relations of C $\{(f,g,h)\in \mathbf{C}[x,y,z]^3 | f^2+g^3+h^6F=0\}$ *has a group structure and it is isomorphic to* $\mathbf{Z}^{2q}$, *where the Alexander polynomial* $\Delta(t)=(t^2-t+1)^q$.

4. New Zariski pair of degree 12. The purpose of this section is to show a new Zariski pair of curves of degree 12. For this purpose, we calculate the Alexander polynomials of three cases which we mentioned in the introduction.

Denote for simplicity

$$\Delta_k = \Delta_k(t) = \left(t-\exp\left(\frac{2k\pi i}{d}\right)\right)\left(t-\exp\left(\frac{-2k\pi i}{d}\right)\right).$$

In particular, when $d=12$, the above polynomials are the following: $\Delta_1=\Delta_{11}=t^2-t\sqrt{3}+1$, $\Delta_2=\Delta_{10}=t^2-t+1$, $\Delta_3=\Delta_9=t^2+1$, $\Delta_4=\Delta_8=t^2+t+1$, $\Delta_5=\Delta_7=t^2+t\sqrt{3}+1$ and $\Delta_6=(t+1)^2$.

4.1. Alexander polynomials of the dual of the Fermat curve and its partner. In Section 2, both of these dual curves have the same singularity configuration $12E_6+16A_1$. According to Lemma 3.2 and Remark 3.3, their Alexander polynomials have the form

$$\Delta(t)=\prod_{k=7}^{11}\Delta_k^{\ell_k} \,\Big|\, \left(\frac{(t^{12}-1)(t-1)}{(t^3-1)(t^4-1)}\right)^{12}.$$

Here $\frac{(t^{12}-1)(t-1)}{(t^3-1)(t^4-1)}=(t^2-t+1)(t^4-t^2+1)$ is the irreducible factorization over $\mathbf{Z}$ of the local Alexander polynomial of the singularity E_6. By Lemma 3.1 and from the above list of Δ_k, we have $\ell_8=\ell_9=0$, $\ell_7=\ell_{11}$ (since $t^4-t^2+1=\Delta_7.\Delta_{11}$). Hence,

$$\Delta(t)=\Delta_7^{\ell_7}.\Delta_{10}^{\ell_{10}}\Delta_{11}^{\ell_{11}}=(t^2-t+1)^{\ell_{10}}(t^4-t^2+1)^{\ell_7}.$$

Thus, we only need to compute ℓ_7 and ℓ_{10}.

The integer ℓ_7 is given by the dimension of the cokernel of

$$\sigma_7 : H^0(\mathbf{P}^2,\mathcal{O}(4))\to \oplus_{P\in Sing}\mathcal{O}_p/\langle u,v\rangle.$$

The left side is the space of quartic curves, hence the dimension is $\frac{5\times 6}{2}=15$, while the right side is the space of dimension 12, since the sum only runs over 12 singularities E_6 (A_1 singularities have trivial $\mathcal{J}_{P,k,d}$, we can omit them). Thus,

$$\ell_7=\dim\operatorname{coker}(\sigma_7)=\dim\ker(\sigma_7)-3.$$

We can use linear algebra to find $\ker(\sigma_7)$ which is the space of quartics passing through 12 E_6 singularities where the coordinates of these singularities are easily find from the given toric decomposition. For the sake of completeness, we list them all here:

- The coordinates of $Sing(F_4^*)$: $(0,\omega,1)$, $(\omega,0,1)$ and $(\omega,1,0)$, where ω runs over the 4th roots of -1.
- The coordinates of $Sing(G_4^*)$: $(1,\pm 1,\pm 2i)$, $(1,\pm 1,\mp 2i)$ and their permutations.

By computation, the dimension of $\ker(\sigma_7)$ in F_4^* case (resp. G_4^* case) is 4 (resp. 3).

Alternatively, using geometrical arguments and the toric decompositions, we can show that the space $\ker(\sigma_7)$ has a basis including $\{x^2yz, xy^2z, xyz^2, x^4+y^4+z^4\}$ (resp. $\{(3x^2+y^2+z^2)(x^2+3y^2+z^2), (3x^2+y^2+z^2)(x^2+y^2+3z^2), (x^2+3y^2+z^2)(x^2+y^2+3z^2)\}$) in F_4^* case (resp. G_4^* case).

Thus, ℓ_7 is 1 (resp. 0) in the case of F_4^* (resp. G_4^*).

The integer ℓ_{10} is given by the dimension of the cokernel of

$$\sigma_{10} : H^0(\mathbf{P}^2,\mathcal{O}(7))\to \oplus_{P\in Sing}\mathcal{O}_p/\langle u,v^2\rangle.$$

Thus

$$\ell_{10}=24-36+\dim\ker(\sigma_{10})=\dim\ker(\sigma_{10})-12.$$

Geometrically, a curve $g_7=0$ of degree 7 is in the kernel of σ_{10}, if it passes through 12 E_6 singularities (locally defined by $u^3+v^4=0$) and it has $v=0$ as the tangent line at each singularity. Using linear algebra, we can find this linear system explicitly, $\dim\ker(\sigma_{10})=13$ in both cases F_4^* and G_4^*.

Furthermore, from the information the $\dim\ker(\sigma_{10})$ and the toric decompositions, we can realize two bases for $\ker(\sigma_{10})$: $\{(xyz)^2l_1, (x^4+y^4+z^4)g_3\}$ (for F_4^* case) and $\{(3x^2+y^2+z^2)(x^2+3y^2+z^2)(x^2+y^2+3z^2)l_1, [7(x^4+y^4+z^4)+18(x^2y^2+y^2z^2+z^2x^2)]g_3\}$ (for G_4^* case), where l_1 and g_3 run over a basis of $H^0(\mathbf{P}^2,\mathcal{O}(1))$ and of $H^0(\mathbf{P}^2,\mathcal{O}(3))$, respectively.

Therefore, in both cases, $\ell_{10}=1$. Thus, we prove the following result.

Proposition 4.1. *The Alexander polynomial of the dual of the Fermat quartic* F_4^* *(resp. the partner curve* G_4^**) is* $(t^2-t+1)(t^4-t^2+1)$ *(resp.* t^2-t+1*). Hence the pair of these two curves is a Zariski pair.*

4.2. Alexander polynomial of the dual of the Klein curve. For the Klein curve, the dual curve has $24A_2+28A_1$ singularities. As mentioned, by Lemma 3.1, the Alexander polynomial has the form

$$\Delta(t) = (t^2 - t + 1)^s,$$

for some non-negative integer s. By Lemma 3.2 and the values of Δ_k, we get $s = \ell_{10}$.

To determine the kernel of the homomorphism

$$\sigma_{10} : H^0(\mathbf{P}^2, \mathcal{O}(7)) \to \oplus_{P \in Sing} \mathcal{O}_p / \langle u, v \rangle$$

the linear algebra technique is hard to follow since the coordinates of $24A_2$ singularities are very complicated. There are two alternative ways to find the kernel of σ_{10}. One can determine a basis of the kernel as $\{(x^3y + y^3z + z^3x)f, (xy^5 + yz^5 + zx^5 - 5x^2y^2z^2)l\}$, where f runs over a basis of $H^0(\mathbf{P}^2, \mathcal{O}(3))$ $(\dim = 10)$ and l runs over a basis of $H^0(\mathbf{P}^2, \mathcal{O}(1))$ $(\dim = 3)$. Thus,

$$\ell_{10} = \dim \operatorname{coker}(\sigma_{10}) = (24 - 36) + 13 = 1.$$

One also can apply Theorem 3.4 to find the same answer. Thus, we complete the proof of the following result.

Proposition 4.2. *The Alexander polynomial of the dual of the Klein curve K_4^* is $t^2 - t + 1$.*

Remark 4.3. The Alexander polynomials of G_4^* and K_4^* are the same as the Alexander polynomial of a generic $(2, 3)$ torus curve of degree 12 (see Theorem 31 [9]).

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