A study of curvature using infinitesimals

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Abstract: Motivated by a profound observation of A'Campo we investigate the behaviour of the curvature of $f(z,w) = c$, $f \in \mathbf{C}\{z,w\}$, $|c|$ small, along infinitesimals. We use the language of infinitesimals as introduced in [2]. Along the way we introduce the important notion of gradient canyon, and prove several theorems in which this notion plays the key role. In this paper we give three such theorems and mention several other facts, to be published elsewhere.

Key words: Newton-Piuseux field; curvature; Lojasiewicz exponent; gradient canyon; Newton-Puiseux infinitesimals.

1. Introduction. Let $f(x,y) \in \mathbf{R}\{x,y\}$ be a real analytic function germ, $f(0,0) = 0$. The level curves $f = c$, $0 < |c| < \epsilon$, have "bumps" near 0, as we all know.

Consider two simple examples:

$$f_2(x,y) = \tfrac{1}{2}x^2 - \tfrac{1}{3}y^3, \quad f_4(x,y) = \tfrac{1}{4}x^4 - \tfrac{1}{5}y^5.$$

We all know $f_2(x,y) = c$ attains maximum curvature when crossing the y-axis. However, a profound observation of N. A'Campo is that this is rather an isolated case. For example, the curvature of $f_4 = c$ is actually 0 on the y-axis; the maximum is attained instead as the level curve crosses $x = \pm ay^{4/3} + \cdots$, $a \neq 0$ a certain constant.

Motivated by this observation we explore this idea using the language of *Newton-Puiseux infinitesimals* ([2], [3], recalled below) and the notion of "*gradient canyon*".

Given $f(z,w) \in \mathbf{C}\{z,w\}$. A level curve $f(z,w) = c$ is a Riemann Surface in $\mathbf{C}^2 = \mathbf{R}^4$, having Gaussian curvature

$$K(z,w) = \frac{2|\Delta_f(z,w)|^2}{(|f_z|^2 + |f_w|^2)^3}, \tag{1.1}$$

$$\Delta_f(z,w) := \begin{vmatrix} f_{zz} & f_{zw} & f_z \\ f_{wz} & f_{ww} & f_w \\ f_z & f_w & 0 \end{vmatrix}.$$

(This is actually the *negative* of the usual Gaussian curvature defined in text books.)

Take a holomorphic map germ

$$\alpha : (\mathbf{C}, 0) \longrightarrow (\mathbf{C}^2, 0), \quad \alpha(t) \not\equiv 0.$$

Let $\alpha_* := Im(\alpha)$ be the image set germ. Being an irreducible curve germ in $\mathbf{C}^2$, it has a unique tangent $T(\alpha_*)$ at 0; $T(\alpha_*)$ is a point of the Riemann Sphere $\mathbf{C}P^1$.

We call α_* a (*Newton-Puiseux*) *infinitesimal* at $T(\alpha_*)$. The *Enriched Riemann Sphere* is $\mathbf{C}P^1_* := \{\alpha_*\}$.

The image of $t \mapsto (at, bt)$ is identified with $[a:b] \in \mathbf{C}P^1$; hence $\mathbf{C}P^1 \subset \mathbf{C}P^1_*$.

The curvature computed *along* α_*, if not zero, can be written as

$$K(\alpha(t)) = as^L + \cdots, \quad a > 0, \quad L \in \mathbf{Q},$$

where $s = s(t)$ is the arc length. This is dominated by the leading term as^L as $s \to 0$.

If $K \equiv 0$ along α_*, we write $(a, L) := (0, \infty)$. Hence we introduce the notations

$$(a, L) := a\delta^L, \quad 0_{\mathcal{V}} := 0\delta^\infty,$$
$$\mathcal{V}(\mathbf{R}) := \{a\delta^L \mid a \neq 0\} \cup \{0_{\mathcal{V}}\},$$

where δ is a symbol.

A *lexicographic ordering* on $\mathcal{V}(\mathbf{R})$ is defined: 0_ν is the smallest element, and

$$a\delta^L > a'\delta^{L'} \textit{ if and only if either } L < L', \textit{ or else } L = L',\ a > a'.$$

Write $a\delta^L \gg a'\delta^{L'}$ (*substantially larger than*) if $L < L'$; *e.g.*, $2\delta^{3/2} > \delta^{3/2} \gg 10^{10}\delta^2 \gg 0_{\mathcal{V}}$.

The *curvature function* K_* on $\mathbf{C}P^1_*$, and the component L_*, are defined as follows:

$$K_* : \mathbf{C}P^1_* \longrightarrow \mathcal{V}(\mathbf{R}), \quad \alpha_* \mapsto a\delta^L; \quad L_*(\alpha_*) := L.$$

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We define the A'Campo bumps of K_*. We tell how to compute them in Theorem A; Theorem B asserts that a bump is a local maximum of K_*. Theorem D gives a formula on the total curvature over a gradient canyon.

A different approach was taken in [1], [4], [5].

To exclude the uninteresting cases, we shall assume *throughout this paper* that $O(f) \geq 2$, and that $K_* \neq const$ on $\mathbf{C}P^1_*$.

For example, $f = (z - w^2)^2$ has $K_* = 8$, a constant. To us, there is no A'Campo bump.

2. Structures on $\mathbf{C}P^1_*$; main results. Recall that the classical Newton-Puiseux Theorem asserts that the field $\mathbf{F}$ of convergent fractional power series in an indeterminate y is algebraically closed. ([6], [7].)

A non-zero element of $\mathbf{F}$ is a (finite or infinite) convergent series

$$(2.1)\qquad \alpha(y) = a_0 y^{n_0/N} + \cdots + a_i y^{n_i/N} + \cdots,\quad n_0 < n_1 < \cdots,\ n_i \in \mathbf{Z},$$

where $0 \neq a_i \in \mathbf{C}$, $N \in \mathbf{Z}^+$, $GCD(N, n_0, n_1, \ldots) = 1$. The *conjugates* of α are

$$\alpha^{(k)}_{conj}(y) := \sum a_i \theta^{kn_i} y^{n_i/N}, \quad 0 \leq k \leq N-1, \qquad \theta := e^{\frac{2\pi\sqrt{-1}}{N}}.$$

The *order* is $O_y(\alpha) := n_0/N$, $O_y(0) := +\infty$.

The following $\mathbf{D}$ is an integral domain with quotient field $\mathbf{F}$, ideals $\mathbf{D}_1$, $\mathbf{D}_{1^+}$:

$$\mathbf{D} := \{\alpha \in \mathbf{F} \mid O_y(\alpha) \geq 0\},\ \mathbf{D}_1 := \{\alpha \mid O_y(\alpha) \geq 1\},\ \mathbf{D}_{1^+} := \{\alpha \mid O_y(\alpha) > 1\}.$$

As in Projective Geometry, $\mathbf{C}P^1_*$ is a union of two charts: $\mathbf{C}P^1_* = \mathbf{C}_* \cup \mathbf{C}'_*$,

$$\mathbf{C}_* := \{\beta_* \in \mathbf{C}P^1_* \mid T(\beta_*) \neq [1:0]\},$$
$$\mathbf{C}'_* := \{\beta_* \in \mathbf{C}P^1_* \mid T(\beta_*) \neq [0:1]\}.$$

Take $\alpha \in \mathbf{D}_1$ as expressed in (2.1). The map germ (abusing notation)

$$\alpha : (\mathbf{C}, 0) \longrightarrow (\mathbf{C}^2, 0), \quad t \mapsto (\alpha(t^N), t^N),$$

is holomorphic. Hence $\alpha_* \in \mathbf{C}_*$; all conjugates of $\alpha(y)$ give the same α_*.

The *Newton-Puiseux coordinate system* on $\mathbf{C}_*$ is, by definition, the surjection

$$\pi : \mathbf{D}_1 \longrightarrow \mathbf{C}_*, \quad \alpha \mapsto \alpha_*.$$

If α_* is given, the congugate class of α is unique; $\alpha \in \mathbf{D}_{1^+}$ if and only if $T(\alpha_*) = [0:1]$.

To define the *contact order* $\mathcal{C}_{ord}(\alpha_*, \beta_*)$, we can of course assume $\alpha_*, \beta_* \in \mathbf{C}_*$. Then $\mathcal{C}_{ord}(\alpha_*, \beta_*)$

$$:= \begin{cases} \infty & \text{if } \alpha_* = \beta_*, \\ \max_{i,j}\{O_y(\alpha^{(i)}_{conj}(y) - \beta^{(j)}_{conj}(y))\} & \text{if } \alpha_* \neq \beta_*. \end{cases}$$

The *horn subspaces* of $\mathbf{C}P^1_*$ centred at α_* of *degree* e, e^+ are, respectively,

$$\mathcal{H}_e(\alpha_*) := \{\beta_* \mid \mathcal{C}_{ord}(\alpha_*, \beta_*) \geq e\},$$
$$\mathcal{H}_{e^+}(\alpha_*) := \{\beta_* \mid \mathcal{C}_{ord}(\alpha_*, \beta_*) > e\}.$$

In particular, $\mathcal{C}_{ord}(\alpha_*, \beta_*) = 1$ if $T(\alpha_*) \neq T(\beta_*)$, and $\mathcal{H}_1(\alpha_*) = \mathbf{C}P^1_*$ for all α_*.

When there is no need to specify α_*, we write $\mathcal{H}_e := \mathcal{H}_e(\alpha_*)$.

We can see that if $\eta(y) = \alpha(y) + [cy^e + \cdots]$, $c \in \mathbf{C}$ *generic*, then $L_*(\eta_*)$ is a constant. We write this constant as $L_*(\mathcal{H}^{grc}_e)$.

A *horn interval* of *radius* r, $r > 0$, is, by definition,

$$\mathcal{H}_e(\alpha_*, r) := \{\beta_* \mid \beta(y) = \alpha(y) + [cy^e + \cdots],\ |c| \leq r\}.$$

Definition 2.1. A horn subspace $\mathcal{H}_e(\alpha_*)$ is a *curvature tableland* if

(1) $\beta_* \in \mathcal{H}_e(\alpha_*) \Longrightarrow L_*(\beta_*) \geq L_*(\mathcal{H}^{grc}_e(\alpha_*))$; and

(2) in the case $e > 1$, there exists e', $1 \leq e' < e$, such that

$$\nu_* \in \mathcal{H}_{e'}(\alpha_*) - \mathcal{H}_e(\alpha_*) \Longrightarrow L_*(\nu_*) > L_*(\mathcal{H}^{grc}_e(\alpha_*)).$$

For $f = z^2 - w^3$, $\mathcal{H}_1$ is *not* a curvature tableland, $L_*(0_*) = -4 < L_*(\mathcal{H}^{grc}_1) = 0$.

Definition 2.2. Let $\mathcal{H}_e$ be a curvature tableland. Take $\beta_* \in \mathcal{H}_e$. We say K_* has an *A'Campo bump* on $\mathcal{H}_{e^+}(\beta_*)$, or simply say $\mathcal{H}_{e^+}(\beta_*)$ is an *A'Campo bump*, if

$$(2.2)\qquad \exists\ \epsilon > 0,\ \mu_* \in \mathcal{H}_e(\beta_*, \epsilon) \Longrightarrow K_*(\beta_*) \geq K_*(\mu_*).$$

Let us apply a unitary transformation (if necessary) so that f is *mini-regular* in z, *i.e.*,

$$(2.3)\qquad f(z,w) := H_m(z,w) + H_{m+1}(z,w) + \cdots,\quad H_m(1,0) \neq 0,$$

where $m = O(f)$, $H_k(z,w)$ a homogeneous form of degree k. Let us also write

$$(2.4)\qquad H_m(z,w) = c(z - c_1 w)^{m_1} \cdots (z - c_r w)^{m_r},\quad m_i \geq 1,\ c_i \neq c_j \text{ if } i \neq j,$$

where $1 \leq r \leq m$, $\sum m_i = m$, $c \neq 0$. Thus $H_m(z,w)$ is *degenerate* if and only if $r < m$.

Let ζ_i denote the Newton-Puiseux roots of $f(z,w)$, and γ_j those of f_z:

$$(2.5)\qquad f(z,w) = unit \cdot \prod_{i=1}^{m}(z-\zeta_i(w)),$$
$$f_z(z,w) = unit \cdot \prod_{j=1}^{m-1}(z-\gamma_j(w)),$$

where $O_w(\zeta_i)$, $O_w(\gamma_j) \geq 1$. Each γ_j, or γ_{j*}, is called a *polar*.

Definition 2.3. Given a polar γ. Let $d_{gr}(\gamma)$ denote the *smallest* number e such that

$$(2.6)\qquad O_w(\|Grad\, f(\gamma(w),w)\|) = O_w(\|Grad\, f(\gamma(w)+uw^e,w)\|),$$

where $u \in \mathbf{C}$ is a *generic* number. We call $d_{gr}(\gamma)$ the *gradient degree* of γ.

The **D**-*gradient canyon* of γ, and the $*$-*gradient canyon* of γ_* are, respectively,

$$\mathcal{G}(\gamma) := \{\alpha \in \mathbf{D}_1 \mid O_y(\alpha-\gamma) \geq d\},$$
$$\mathcal{G}_*(\gamma_*) := \mathcal{H}_d(\gamma_*), \quad d := d_{gr}(\gamma).$$

When there is no confusion, we write

$$d := d_{gr}(\gamma), \quad \mathcal{G} := \mathcal{G}(\gamma), \quad \mathcal{G}_* := \mathcal{G}_*(\gamma_*).$$

The *degree* and *multiplicity* of $\mathcal{G}$, $\mathcal{G}_*$ are, respectively,

$$d_{gr}(\mathcal{G}) := d_{gr}(\mathcal{G}_*) := d_{gr}(\gamma),$$
$$m(\mathcal{G}) := m(\mathcal{G}_*) := \sharp\{k | \mathcal{G}(\gamma_k) = \mathcal{G}(\gamma)\}.$$

Note. In general, $m(\mathcal{G}) \neq \sharp\{k|\gamma_k \in \mathcal{G}(\gamma)\}$. See Example 2.5.

Definition 2.4. We say γ is *maximal*, or has *maximal gradient degree*, if

(1) $d_{gr}(\gamma) < \infty$,

(2) $O(\gamma_j-\gamma) \geq d_{gr}(\gamma) \Longrightarrow d_{gr}(\gamma_j) = d_{gr}(\gamma)$, *i.e.*, $\mathcal{G}(\gamma_j) \subseteq \mathcal{G}(\gamma) \Longrightarrow \mathcal{G}(\gamma_j) = \mathcal{G}(\gamma)$.

In this case we say the **D**-gradient canyon $\mathcal{G}(\gamma)$ is *minimal*.

Note. If γ is a multiple root of $f(z,w)$, then $d=\infty$, $\mathcal{G}(\gamma) = \{\gamma\}$.

Example 2.5. Take $f = z^m - w^n$, $2 \leq m \leq n$. There is only one polar $\gamma = 0$,

$$d_{gr}(\gamma) = \tfrac{n-1}{m-1}, \quad \mathcal{G} = \{uy^{\frac{n-1}{m-1}} + \cdots \mid u \in \mathbf{C}\},$$
$$m(\mathcal{G}) = m-1.$$

Take $g = z^4 - 2z^2w^2 - w^{100}$, $\gamma_1 = 0$, $\gamma_2, \gamma_3 = \pm w$. Then $d_{gr}(\gamma_1) = 97$, $d_{gr}(\gamma_2) = 1$,

$$\mathcal{G}(\gamma_1) \subset \mathcal{G}(\gamma_2) = \mathcal{G}(\gamma_3) = \mathbf{D}_1,$$
$$m(\mathcal{G}(\gamma_2)) = 2, \quad \sharp\{k|\gamma_k \in \mathcal{G}(\gamma_2)\} = 3.$$

Here γ_1 is maximal, but γ_2, γ_3 are not.

Given γ. We now define $L_\gamma \in \mathbf{Q}$, and a rational function $R_\gamma(u)$, $R_\gamma(u) \geq 0$, $u \in \mathbf{C}$.

We can assume $\gamma \in \mathbf{D}_{1^+}$ so that $T(\gamma_*) = [0:1]$. If $d>1$, define L_γ, $R_\gamma(u)$ by

$$(2.7)\qquad K(\gamma(y)+uy^d, y) := 2R_\gamma(u)y^{2L_\gamma} + \cdots, \quad R_\gamma(u) \not\equiv 0,$$

where y can be considered as the arc length of $(\gamma(y)+uy^d)_*$ since $\lim y/s = 1$.

In the case $d=1$, L_γ and $R_\gamma(u)$ are defined by

$$(2.8)\quad K\Big(\gamma(y)+\tfrac{uy}{\sqrt{1+|u|^2}}, \tfrac{y}{\sqrt{1+|u|^2}}\Big) := 2R_\gamma(u)y^{2L_\gamma} + \cdots, \quad R_\gamma(u) \not\equiv 0.$$

Lemma 2.6. *The function $R_\gamma(u) \geq 0$ is defined and continuous for all $u \in \mathbf{C}$,*

$$(2.9)\qquad L_\gamma = -d, \quad \lim_{u\to\infty} R_\gamma(u) = 0.$$

Hence the absolute maximum of $R_\gamma(u)$ is attained. (There may be many local maxima.)

Theorem A. *A minimal* **D**-*gradient canyon is a curvature tableland, and vice versa.*

Take a maximal polar γ and a local maximum $R_\gamma(c)$ of $R_\gamma(u)$. Then $\mathcal{H}_{d^+}(\gamma_*^{+c})$ is an A'Campo bump, where $\gamma^{+c}(y) := \gamma(y)+cy^d$. All A'Campo bumps can be found in this way.

No A'Campo bump arises from a multiple root of $f(z,w)$. We can ignore such polars.

Example 2.7. For $f_2(z,w) = \frac{1}{2}z^2 - \frac{1}{3}w^3$, there is only one polar $\gamma = 0$, having $d=2$,

$$R_\gamma(u) = (|u|^2+1)^{-3}, \quad K_*((\gamma+uy^d)_*) = 2R_\gamma(u)\delta^{-4}.$$

In this example, $R_\gamma(u)$ is maximum at $u=0$.

Next, consider $f_4(z,w) = \frac{1}{4}z^4 - \frac{1}{5}w^5$, having $\gamma = 0$, $d = \frac{4}{3}$,

$$\Delta = z^2w^3(4z^4-3w^5), \quad R_\gamma(u) = 9|u|^4(|u|^6+1)^{-3}.$$

Here $R_\gamma(0) = 0$, a minimum; $R_\gamma(u)$ attains maximum on the circle $|u| = (2/7)^{1/6}$.

We now define the *perturbation topology* on $\mathcal{V}(\mathbf{R})$: a closed set is $\mathcal{V}(\mathbf{R})$, or $\emptyset$, or

$$C_{\varepsilon,\sigma} := \{a_1\delta^{q_1} \mid \epsilon_1 \leq a_1 < \infty\} \cup \cdots \cup \{a_s\delta^{q_s} \mid \epsilon_s \leq a_s < \infty\},$$

where $\varepsilon := \{\epsilon_1,\ldots,\epsilon_s\}$, $\epsilon_i > 0$, $\sigma := \{q_1,\ldots,q_s\} \subset \mathbf{Q}$. (The closure of $0_{\mathcal{V}}$ is the whole space $\mathcal{V}(\mathbf{R})$.)

The *perturbation topology* on $\mathbf{C}_*$ is the induced one. A neighbourhood of μ_* is

$$\mathcal{N}^{ptb}_{\varepsilon,\sigma}(\mu_*) := \{\nu_* \mid \nu(y)-\mu(y) = ay^q + \cdots,\ |a|\delta^q \notin C_{\varepsilon,\sigma}\}.$$

The topology on $\mathbf{C}'_*$ is similarly defined; that on $\mathbf{C}P^1_*$ is generated by these two.

Theorem B. *Every μ_* in an A'Campo bump $\mathcal{H}_{d^+}(\gamma_*^{+c})$ is a local maximum of K_* in the perturbation topology.*

Theorem C. *Suppose* $1 < d := d_{gc}(\gamma) < \infty$. *Then*

$$\begin{aligned}(2.10)\qquad \int_{\mathcal{G}} K &:= 2\Big\{\int_{\mathbf{R}^2} R_\gamma(u)\, dx \wedge dy\Big\}\,\delta^{2L_\gamma}\\ &= m(\mathcal{G})\pi\,\delta^{-2d}, \quad \mathcal{G} := \mathcal{G}(\gamma),\end{aligned}$$

where $u = x + iy \in \mathbf{C}$. The integral is called the total Gaussian curvature over $\mathcal{G}$.

3. The Lojasiewicz exponent function $L_\gamma(e)$. Let γ be given. Take $e \geq 1$, $u \in \mathbf{C}$ (or an *indeterminate*). Write

$$\begin{aligned}(3.1)\quad &|\Delta_f(\gamma(y) + uy^e, y)|^2\\ &\qquad := N_{(\gamma,e)}(u)y^{2L_\Delta(\gamma,e)} + \cdots, \quad N_{(\gamma,e)}(u) \not\equiv 0,\\ &\|\mathit{Grad}\, f(\gamma(y) + uy^e, y)\|^2\\ &\qquad := D_{(\gamma,e)}(u)y^{2L_{Grad}(\gamma,e)} + \cdots, \ D_{(\gamma,e)}(u) \not\equiv 0,\end{aligned}$$

where $N_{(\gamma,e)}(u)$, $D_{(\gamma,e)}(u)$ are real-valued, non-negative, polynomials of u, $\bar{u}$. We define

$$\begin{aligned}(3.2)\qquad L_\gamma(e) &:= L_\Delta(\gamma, e) - 3L_{Grad}(\gamma, e),\\ R_{(\gamma,e)}(u) &:= N_{(\gamma,e)}(u)D_{(\gamma,e)}(u)^{-3}.\end{aligned}$$

Note that $L_\Delta(\gamma, e)$, $L_{Grad}(\gamma, e)$ are also defined when e is *irrational*. Thus, when γ is fixed, these are *piece-wise linear, continuous, increasing* functions of e, $1 \leq e < \infty$.

As in Calculus, we say $\phi(x)$ is *increasing* (resp. *decreasing*, resp. *strictly decreasing*) if

$$\begin{aligned}&x_1 < x_2 \Longrightarrow \phi(x_1) \leq \phi(x_2)\\ &\quad(\text{resp. } \phi(x_1) \geq \phi(x_2), \text{ resp. } \phi(x_1) > \phi(x_2)).\end{aligned}$$

Now let γ be a given polar. Note that $R_{(\gamma,d)}(u) = R_\gamma(u)$ in (2.7). We write

$$\begin{aligned}(3.3)\qquad \mathcal{C}_f(\gamma) &:= \mathcal{C}_f(\gamma_*)\\ &:= \max\{O_y(\gamma - \zeta_i) \mid 1 \leq i \leq m\},\\ &\qquad\qquad \zeta_i \text{ in (2.5)}.\end{aligned}$$

Lemma 3.1. *If $1 < d < \infty$, then $L_\gamma(d) = -d = L_\gamma$, L_γ being the constant defined in* (2.7).

If $d = 1$, then $\mathcal{C}_f(\gamma) = 1$, $L_\gamma(1) = -1 = L_\gamma$.

4. Newton polygon relative to a polar. Let γ be a given polar, not a multiple root of $f(z, w)$ (fixed in what follows), *i.e.*, $f(\gamma(w), w) \neq 0$. We can apply a unitary transformation, if necessary, so that $T(\gamma_*) = [0 : 1]$, $\gamma \in \mathbf{D}_{1^+}$.

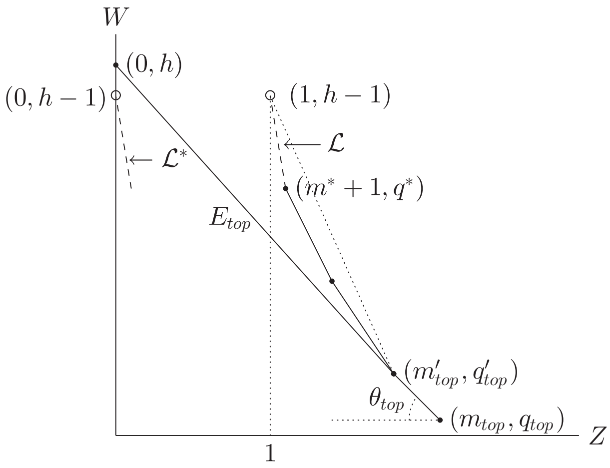

Fig. 1. $\mathcal{NP}(F)$, $\mathcal{NP}(F_Z)$.

Let us change coordinates:

$$\begin{aligned}(4.1)\qquad &Z := z - \gamma(w), \quad W := w,\\ &F(Z, W) := f(Z + \gamma(W), W),\end{aligned}$$

and write $A \approx B$ when $A/B \to 1$, then

$$\begin{aligned}(4.2)\qquad &\Delta_f(z, w) = \Delta_F(Z, W) + \gamma''(W)F_Z^3,\\ &\|\mathit{Grad}_{z,w} f\| \approx \|\mathit{Grad}_{Z,W} F\|.\end{aligned}$$

An important step is to study the relationship between the Newton Polygons $\mathcal{NP}(F)$ and $\mathcal{NP}(F_Z)$. This is illustrated in Fig. 1, which is deliberately drawn off scale for clarity; a number of key arguments are also exposed.

Recall that a monomial term aZ^iW^q, $a \neq 0$, $q \in \mathbf{Q}$, is represented by a "Newton dot" at (i, q). We shall simply say (i, q) is a *dot.*

If $i \geq 1$, then (i, q) is a dot of $F(Z, W)$ if and only if $(i - 1, q)$ is one of F_Z. Since γ is a polar, F_Z *has no dot of the form* $(0, q)$; $F(Z, W)$ has no dot of the form $(1, q)$.

As $f(\gamma(w), w) \neq 0$, we know $F(0, W) \neq 0$. Hence

$$\begin{aligned}(4.3)\qquad &F(0, W) := aW^h + \cdots, \quad a \neq 0,\\ &h := O_W(F(0, W)),\end{aligned}$$

and then $(0, h)$ is a vertex of $\mathcal{NP}(F)$, $(0, h - 1)$ is one of $\mathcal{NP}(F_W)$.

Let E_{top} denote the top edge of $\mathcal{NP}(F)$, *i.e.*, the edge with left vertex $(0, h)$. Let (m_{top}, q_{top}) denote the right vertex of E_{top}, and θ_{top} the angle of E_{top}, as shown in Fig. 1,

$$\tan \theta_{top} = \text{co-slope of } E_{top},$$

where the *co-slope* of a line passing through $(x, 0)$ and $(0, y)$ is, by definition, y/x.

Notations. Let a weight system $\omega = (\omega_Z, 1)$ be given. The weighted initial form of $G(Z, W)$ (in the weighted Taylor expansion) is denoted by $\mathcal{I}_\omega(G)(Z, W)$, or simply $\mathcal{I}_\omega(G)$.

Let $\mathcal{I}_\omega(G) := \sum a_{ij} Z^i W^{j/N}$. The weighted order of G is $O_\omega(G) := i\omega_Z + j/N$.

As usual, the degree of a polynomial $P(Z)$ is written as $\deg P(Z)$.

In Fig. 1, $\mathcal{L}^*$ denotes the line joining $(0, h-1)$ (which is not a dot of F_Z) and a dot of F_Z such that no dot of F_Z lies below $\mathcal{L}^*$.

No dot of F_W lies below $\mathcal{L}^*$; if $d_{gr}(\gamma) > 1$, $(0, h-1)$ is the only dot of F_W lying on $\mathcal{L}^*$.

Finally, the line, $\mathcal{L}$, through $(1, h-1)$ is parallel to $\mathcal{L}^*$ and its co-slope is precisely $d_{gr}(\gamma)$.

For $f(z, w)$ in Example (2.5), $\mathcal{L}^*$ is the line joining $(0, n-1)$, $(m-1, 0)$; $d_{gr}(\gamma) = \frac{n-1}{m-1}$.

Scketch of Proof of Theorem C. We already know $L_\gamma(d) = -d$ from Lemma 3.1. □

Let $\omega = (d, 1)$. Write $p(u) := \mathcal{I}_\omega(F_Z)(u, 1)$, $c := |ha|$, and $\mathcal{G} := \mathcal{G}(\gamma)$. Note that

$$\deg p(u) = m(\mathcal{G}),\ N_{(\gamma,d)}(u) = c^4 |p'(u)|^2,$$
$$D_{(\gamma,d)}(u) = c^2 + |p(u)|^2.$$

Now let us write

$$p(u) := U(x, y) + iV(x, y), \quad u = x + iy \in \mathbf{C},$$

where U, V satisfy the Cauchy-Riemann equations. Using the latter we find

$$\begin{aligned}\int R_\gamma(u) du \wedge d\bar{u} &= \int \frac{c^4 |p'(u)|^2}{[c^2 + |p(u)|^2]^3}\, du \wedge d\bar{u} \\ &= -2i \int \frac{dU \wedge dV}{[1 + U^2 + V^2]^3}.\end{aligned}$$

The mapping $u \mapsto p(u)$ is a $m(\mathcal{G})$-sheet branch covering of $\mathbf{C}$. Hence

$$\begin{aligned}\int_{\mathbf{C}} R_\gamma(u) du \wedge d\bar{u} &= -2im(\mathcal{G}) \int_{\mathbf{R}^2} \frac{dU \wedge dV}{[1 + U^2 + V^2]^3} \\ &= -\pi i m(\mathcal{G}),\end{aligned}$$

and (2.10) follows from the identity $du \wedge d\bar{u} = -2i dx \wedge dy$.

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