An area minimizing scheme for anisotropic mean curvature flow

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(Communicated by Masaki KASHIWARA, M.J.A., Dec. 12, 2011)

Abstract: We consider an area minimizing scheme for anisotropic mean curvature flow originally due to Chambolle (2004). We show the convergence of the scheme to anisotropic mean curvature flow in the sense of Hausdorff distance by the level set method provided that no fattening occurs.

Key words: Anisotropic mean curvature flow; approximation scheme; area minimization; viscosity solutions.

1. Introduction. We provide a variational approximation scheme to the anisotropic mean curvature flow; a family $\{\Gamma_t\}_{t\geq 0}$ of closed hypersurfaces in $\mathbf{R}^N$ evolving by the following equation

$$V = -\gamma(\mathbf{n})\mathrm{div}_{\Gamma_t}\xi(\mathbf{n}). \tag{1}$$

Here $\mathbf{n}$ is the outer unit normal vector field of Γ_t, V is the normal velocity of Γ_t in the direction of $\mathbf{n}$, div_{Γ_t} denotes the surface divergence on Γ_t, $\gamma = \gamma(p)$ is a *surface energy density* and $\xi = \nabla\gamma$ is called the Cahn-Hoffman vector. We assume that γ is nonnegative, convex, even, positively homogeneous of degree 1 and satisfies $\lambda|p| \leq \gamma(p) \leq \Lambda|p|$ for $p \in \mathbf{R}^N$ and some $0 < \lambda \leq \Lambda < +\infty$. Such γ's are called anisotropy. Besides, we assume that $\gamma \in C^2(\mathbf{R}^N\backslash\{0\})$.

As well known, many people studied various algorithm to compute the mean curvature flow (MCF for short). Among them, Almgren-Taylor-Wang [1] introduced a variational approach to constructing the anisotropic MCF. It is based on a time-discretization on a minimization problem for computing the surface. However, the main drawback of their approach is the lack of the uniqueness of the minimizer to their variational problem. To resolve this uniqueness problem, Chambolle [6] proposed another approach which provides a monotonous selection of the discrete evolution of [1]. He also proved in [6] the convergence of his scheme to the MCF in L^1-topology, whenever no fattening occurs.

In this note we provide a new scheme via Chambolle's one [6] and show the locally uniform convergence of our scheme to the level set equation for (1) (cf. Giga [11]). This yields the convergence of the level set for our scheme to the level set flow for (1) in the sense of the Hausdorff distance, whenever no fattening occurs. Notice that Eto [10] essentially already obtained such results in the case where $N = 2$ and $\gamma(p) = |p|$ (isotropic case) although a complete proof is not given. Thus our results extend [10] to the case where $N \geq 2$ and γ is anisotropic and are sharper than that of [6].

In the following of this note we state our main results and some key ingredients which are not explicit in [10] even for $\gamma(p) = |p|$ almost without proofs; the details will be published elsewhere. Recently, we learned that in [7] and [8] Chambolle and Novaga considered approximation schemes to (1), which are the anisotropic versions of [3] and [6]. However, their results and proofs are different from ours.

2. Preliminaries. Let γ be a surface energy density satisfying the same assumption as in section 1. The dual function γ° for γ is defined by $\gamma^\circ(p) := \sup_{\gamma(q)\leq 1}\langle p, q\rangle$. In the following of this note, we make the following assumptions in addition to those in the previous section:

$$\gamma, \gamma^\circ \in C^2(\mathbf{R}^N\backslash\{0\}),$$
$$\nabla^2\gamma^2, \nabla^2(\gamma^\circ)^2 > O \quad \text{in } \mathbf{R}^N\backslash\{0\}.$$

2000 Mathematics Subject Classification. Primary 35D40, 35K65, 53C44; Secondary 35K65, 35K67.

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doi: 10.3792/pjaa.88.7

We denote by $\partial\gamma(p)$ the subdifferential of γ at $p \in \mathbf{R}^N$. For a set $E \subset \mathbf{R}^N$, we define the anisotropic signed distance function d_E to ∂E by

$$d_E(x) := \inf_{y\in E} \gamma^\circ(x-y) - \inf_{y\in \mathbf{R}^N\setminus E} \gamma^\circ(x-y).$$

Then $\nabla d_E(x) = \mathbf{n}(x)/\gamma^\circ(\mathbf{n}(x))$ for all $x \in \partial E$ where ∇d_E exists. Moreover, the anisotropic mean curvature $\kappa_E(x)$ at $x \in \partial E$ is defined by

$$\kappa_E(x) := -\mathrm{div}_{\Gamma_t}\xi(\mathbf{n}(x))$$

for $x \in \partial E$ whenever $\nabla^2 d_E(x)$ exists.

Let Ω be an open subset of $\mathbf{R}^N$. We say a function $u \in L^1(\Omega)$ is a function of bounded variation, denoting by $u \in BV(\Omega)$, if its distributional gradient Du is a (vector-valued) Radon measure. For any $u \in BV(\Omega)$, we define the anisotropic total variation of u with respect to γ in Ω by

$$\int_\Omega \gamma(Du) := \sup_{\varphi\in\mathcal{D}_\gamma} \int_\Omega u \,\mathrm{div}\,\varphi\, dx,$$

$$\mathcal{D}_\gamma := \{\varphi \in C_0^1(\Omega;\mathbf{R}^N) \mid \gamma^\circ(\varphi) \le 1 \text{ in } \Omega\}.$$

(cf. [2]).

For any $g \in L^2_{loc}(\mathbf{R}^N)$ and $h > 0$, we consider the following elliptic inclusion:

$$(2) \qquad u - h\,\mathrm{div}\,\partial\gamma(\nabla u) \ni g \quad \text{in } \mathbf{R}^N.$$

This equation can be derived as the Euler-Lagrange equation for the minimization problem:

$$(3) \quad J(u) = \min_{v\in L^2(\Omega)} J(v) \text{ for } u \in L^2(\Omega)\cap BV(\Omega),$$

$$J(v) := \begin{cases} \displaystyle\int_\Omega \gamma(Dv) + \frac{1}{2h}\|v-g\|^2_{L^2(\Omega)} \\ \qquad \text{if } v \in L^2(\Omega)\cap BV(\Omega), \\ +\infty \\ \qquad \text{if } v \in L^2(\Omega)\setminus BV(\Omega). \end{cases}$$

We say $u \in L^2_{loc}(\mathbf{R}^N) \cap BV_{loc}(\mathbf{R}^N)$ is a solution of (2), provided that there exists a vector field $z \in L^\infty(\mathbf{R}^N;\mathbf{R}^N)$, $\mathrm{div}\, z \in L^2_{loc}(\mathbf{R}^N)$ satisfying

$$z(x) \in \partial\gamma(\nabla u(x)) \text{ a.e. } x \in \mathbf{R}^N,$$

$$(z, Du) = \gamma(Du) \text{ locally as measures in } \mathbf{R}^N,$$

$$u - h\,\mathrm{div}\, z = g \text{ in } \mathcal{D}'(\mathbf{R}^N).$$

Here $BV_{loc}(\mathbf{R}^N) := \{u \in L^1_{loc}(\mathbf{R}^N) \mid u \in BV(K)$ for any compact set $K \subset \mathbf{R}^N\}$. Applying [6], [2], and [15], one is able to show the existence and uniqueness of solutions of (2). Even in a rigorous sense (2) is the Euler-Lagrange equation for the minimization problem (3). Indeed, we observe that u is a solution of (2) if and only if u satisfies the following variational inequality: For any $R > 0$ and $\phi \in C_0^1(B(0,R))$,

$$\int_{B(0,R)} \gamma(Du) + \frac{1}{2h}\|u-g\|^2_{L^2(B(0,R))}$$
$$\le \int_{B(0,R)} \gamma(D(u+\phi))$$
$$+ \frac{1}{2h}\|(u+\phi)-g\|^2_{L^2(B(0,R))}.$$

See [5] and [15] for the detail.

Let E be a closed set in $\mathbf{R}^N$ and let d_E be the anisotropic signed distance function. We then observe by the same argument as in [6], [5] and [10] that there is a unique solution $u \in C(\mathbf{R}^N) \cap BV_{loc}(\mathbf{R}^N)$ of (2) satisfying $\|\nabla u\|_{L^\infty(\mathbf{R}^N)} < +\infty$.

3. Chambolle's version of the Almgren-Taylor-Wang scheme. Let E_0 be a closed subset of $\mathbf{R}^N$ and fix a time step $h > 0$. Let $w^{E_0}_{\gamma^\circ,h}$ be a unique solution of (2) with $g = d_{E_0}$. We set

$$T_{\gamma^\circ,h}(E_0) := \{x \in \mathbf{R}^N \mid w^{E_0}_{\gamma^\circ,h}(x) \le 0\},$$

$$E^h_t := T^{[t/h]}_{\gamma^\circ,h}(E_0) \quad \text{for } t \in [0,T),$$

where $[t/h]$ is the integer part of t/h and $T^k_{\gamma^\circ,h}(E_0) := T_{\gamma^\circ,h}(T^{k-1}_{\gamma^\circ,h}(E_0))$ for $k \in \mathbf{N}$. Letting $h \to 0$, we formally obtain in the limit a flow $\{E_t\}_{t\ge 0}$ of a closed subset of $\mathbf{R}^N$ whose boundary evolves by (1). This is confirmed in measure theoretic sense (L^1 sense) by Chambolle [6].

4. Set operator and function operator. Denote by $\mathcal{C}(\mathbf{R}^N)$ the family of all closed subsets of $\mathbf{R}^N$. Let $T_{\gamma^\circ,h}$ be the set operator defined in the previous section. Then we observe that $T_{\gamma^\circ,h}$ maps from $\mathcal{C}(\mathbf{R}^N)$ to $\mathcal{C}(\mathbf{R}^N)$ and that it satisfies the following properties: For E, E', $E_n \in \mathcal{C}(\mathbf{R}^N)$, $x \in \mathbf{R}^N$ and $u \in UC(\mathbf{R}^N)$,

$$T_{\gamma^\circ,h}(E) \subset T_{\gamma^\circ,h}(E') \text{ if } E \subset E',$$

$$T_{\gamma^\circ,h}(E_n) \searrow T_{\gamma^\circ,h}(E) \text{ if } E_n \searrow E,$$

$$T_{\gamma^\circ,h}(x+E) = x + T_{\gamma^\circ,h}(E),$$

$$\bigcap_{\mu\in\mathbf{R}} T_{\gamma^\circ,h}([u \ge \mu]) = \emptyset,$$

$$\bigcup_{\mu\in\mathbf{R}} T_{\gamma^\circ,h}([u \ge \mu]) = \mathbf{R}^N.$$

Here $UC(\mathbf{R}^N)$ is the class of all uniformly continuous functions on $\mathbf{R}^N$.

Following Matheron [14] and Cao [4], we convert the set operator $T_{\gamma^\circ,h}$ to a function operator $S_{\gamma^\circ,h}$ in the following way: For any $u \in C(\mathbf{R}^N)$,

(4) $[S_{\gamma^\circ,h}u](x) := \sup\{\mu \in \mathbf{R} \mid x \in T_{\gamma^\circ,h}([u \ge \mu])\},$
$[u \ge \mu] := \{x \in \mathbf{R}^N \mid u(x) \ge \mu\}.$

Then we observe from the properties of $T_{\gamma^\circ,h}$ mentioned above that $S_{\gamma^\circ,h}$ maps from $UC(\mathbf{R}^N)$ to $UC(\mathbf{R}^N)$ and that for any u, $v \in UC(\mathbf{R}^N)$, $c \in \mathbf{R}^N$ and nondecreasing $g \in C(\mathbf{R})$,

$$S_{\gamma^\circ,h}u \le S_{\gamma^\circ,h}v \text{ in } \mathbf{R}^N \text{ if } u \le v \text{ in } \mathbf{R}^N,$$
$$[S_{\gamma^\circ,h}u](x+c) = [S_{\gamma^\circ,h}u(\cdot + c)](x) \text{ for all } x \in \mathbf{R}^N,$$
$$S_{\gamma^\circ,h}(g \circ u) = g \circ S_{\gamma^\circ,h}u.$$

Moreover, applying a theorem (for morphological operators) due to [14] (cf. [4]), we obtain the sup-inf representation for $S_{\gamma^\circ,h}$: For $u \in C(\mathbf{R}^N)$ and $h > 0$,

$$[S_{\gamma^\circ,h}u](x) = \sup_{X \in \mathcal{B}_{\gamma^\circ,h}} \inf_{y \in X} u(x+y) \tag{5}$$

for all $x \in \mathbf{R}^N$, where $\mathcal{B}_{\gamma^\circ,h} := \{X \in \mathcal{C}(\mathbf{R}^N) \mid [S_{\gamma^\circ,h}(-d_X)](0) \ge 0\}$. We use this formula to derive the generator of $S_{\gamma^\circ,h}$ in the next section.

5. Key estimates. We obtain that the supremum in (5) is actually achieved. A precise form of our assertion is stated below. Set $r_1 := 2N/\sqrt{N+1}$ and $U_1 := \{y \in \mathbf{R}^N \mid \gamma^\circ(y) \le r_1\sqrt{h}\}$.

Proposition 5.1. *Let $u \in C(\mathbf{R}^N)$. For any $h > 0$ and $x \in \mathbf{R}^N$, there exists $X_1 \in \mathcal{B}_{\gamma^\circ,h}$ such that*

$$[S_{\gamma^\circ,h}u](x) = \inf_{y \in X_1 \cap U_1} u(x+y). \tag{6}$$

Let $\psi \in C^\infty(\mathbf{R}^N)$ and assume that $\psi(0) = 0$ and $\nabla\psi(0) = |\nabla\psi(0)|e_N$ $(\neq 0)$ $(e_N := (0, \cdots, 0, 1))$. Then there is $\delta > 0$ such that for all $y \in B(0, 4\delta)$,

$$\frac{1}{2}|\nabla\psi(0)| \le |\nabla\psi(y)| \le 2|\nabla\psi(0)|. \tag{7}$$

Let $X_1 \in \mathcal{B}_{\gamma^\circ,h}$ satisfy (6) with $u = \psi$ and $x = 0$. We get from (6) and (7)

$$X_1 = [\psi \ge [S_{\gamma^\circ,h}\psi](0)], \ X_1 \cap U_1 \neq \emptyset.$$

From (7) and this, we have

$$|[S_{\gamma^\circ,h}\psi](0)| \le 2|\nabla\psi(0)|r_1\sqrt{h}. \tag{8}$$

Using suitable barrier functions and the comparison principle for (2), we are able to approximate $w^E_{\gamma^\circ,h}$ by the anisotropic mean curvature of a closed set E as follows.

Proposition 5.2. *Let $\psi \in C^\infty(\mathbf{R}^N)$ satisfy the above assumption and take $\delta > 0$ so that (7) holds. Set $E := [\psi \ge \mu]$. For each $h > 0$, let $w^E_{\gamma^\circ,h}$ be a unique solution of (2) with $g = d_E$. Then for any $\varepsilon > 0$, there exist $r \in (0, \delta)$ and $h_1 > 0$ such that*

$$|w^E_{\gamma^\circ,h} - (d_E + h\kappa_E)| \le \varepsilon h \quad \text{on } \overline{B(0,r)} \tag{9}$$

for all $h \in (0, h_1)$.

By Proposition 5.2 we may assume $|\mu| \le C_1 h$ for some $C_1 > 0$ independent of $h > 0$. We observe by the anisotropy of γ that $|d_E(0) - \mu/\gamma(\nabla\psi(0))| \le C_2 h^2$ for some $C_2 > 0$ independent of small $h > 0$. Thus applying (9) and this estimate, we obtain the following estimate for $[S_{\gamma^\circ,h}\psi](0)$.

Theorem 5.1. *Let ψ and E be the same as in Proposition 5.2. Then for any $\varepsilon > 0$, there is $h_2 > 0$ such that*

$$|[S_{\gamma^\circ,h}\psi](0) - h\gamma(\nabla\psi(0))\kappa_E(0)| \le \varepsilon h$$

for all $h \in (0, h_2)$.

Let $\phi \in C^\infty(\mathbf{R}^N)$ and $z \in \mathbf{R}^N$ satisfy $\nabla\phi(z) \neq 0$. Since we easily see that $[S_{\gamma^\circ,h}\phi](x) = \phi(x) + [S_{\gamma^\circ,h}\psi](0)$ $(\psi(y) := \phi(x+y) - \phi(x))$ for all x close to z, we use Theorem 5.1 to obtain the generator of $S_{\gamma^\circ,h}$.

Theorem 5.2. *Let $\phi \in C^\infty(\mathbf{R}^N)$, $z \in \mathbf{R}^N$ and $\varepsilon > 0$. Then if $\nabla\phi(z) \neq 0$, then there exist $\delta > 0$ and $h_0 > 0$ such that for all $x \in B(z, \delta)$ and $h \in (0, h_0)$,*

$$[S_{\gamma^\circ,h}\phi](x) \le \phi(x) + \{-F(\nabla\phi(x), \nabla^2\phi(x)) + \varepsilon\}h,$$
$$[S_{\gamma^\circ,h}\phi](x) \ge \phi(x) + \{-F(\nabla\phi(x), \nabla^2\phi(x)) - \varepsilon\}h,$$

where $F(p, X) := -\gamma(p)\ \mathrm{tr}(\nabla^2\gamma(p)X)$.

We have mentioned in section 4 that $S_{\gamma^\circ,h}$ maps from $UC(\mathbf{R}^N)$ to $UC(\mathbf{R}^N)$. As for this fact, we have the following estimate.

Proposition 5.3. *For $u_0 \in UC(\mathbf{R}^N)$, we have*

$$|[S_{\gamma^\circ,h}u_0](x) - [S_{\gamma^\circ,h}u_0](y)| \le \omega_0(|x-y|)$$

for all x, $y \in \mathbf{R}^N$, where ω_0 denotes the modulus of continuity of u_0.

6. Convergence. Let $u_0 \in UC(\mathbf{R}^N)$. For $t \ge 0$ and $x \in \mathbf{R}^N$, define

$$u^h(t,x) := [S^{[t/h]}_{\gamma^\circ,h}u_0](x),$$
$$\overline{u}(t,x) := \limsup_{\substack{(s,y)\to(t,x)\\ h\to 0}} u^h(s,y),$$
$$\underline{u}(t,x) := \liminf_{\substack{(s,y)\to(t,x)\\ h\to 0}} u^h(s,y).$$

Here $[S^k_{\gamma^\circ,h}u_0](x) := [S_{\gamma^\circ,h}[S^{k-1}_{\gamma^\circ,h}u_0]](x)$ for $k \in \mathbf{N}$. We then verify by Theorem 5.2 that $\overline{u}$ (resp., $\underline{u}$) is a viscosity subsolution (resp., supersolution) of

$$u_t + F(\nabla u, \nabla^2 u) = 0 \quad \text{in } (0,T) \times \mathbf{R}^N, \tag{10}$$

and by Proposition 5.3 that they satisfy $\overline{u}(0,x) = \underline{u}(0,x) = u_0(x)$ for $x \in \mathbf{R}^N$ (cf. [13], [11] and [10]).

Applying the comparison principle and the stability for viscosity solutions (cf. [12], [9], [13] and [11]), we obtain the following convergence result.

Theorem 6.1. *Assume* $u_0 \in UC(\mathbf{R}^N)$. *Then as* $h \to 0$, u^h *converges to* $u \in C([0,T) \times \mathbf{R}^N)$ *locally uniformly in* $[0,T) \times \mathbf{R}^N$ *and* u *is a unique viscosity solution of* (10) *satisfying* $u(0,x) = u_0(x)$ *for* $x \in \mathbf{R}^N$.

Let E_0 be a compact set in $\mathbf{R}^N$. We choose $u_0 \in UC(\mathbf{R}^N)$ satisfying $u_0 > 0$ in $\operatorname{int} E_0$, $u_0 = 0$ on ∂E_0 and $u_0 < 0$ in $\mathbf{R}^N \backslash E_0$. Then we see by [10] that $[S^{[t/h]}_{\gamma^\circ,h} u_0 \geq 0] = T^{[t/h]}_{\gamma^\circ,h}(E_0) = E^h_t$. Set $E_t := [u(t,\cdot) \geq 0]$, where u is a unique viscosity solution of (10) satisfying $u(0,x) = u_0(x)$ for $x \in \mathbf{R}^N$. By use of Theorem 6.1, we establish the convergence of the discrete evolution $\{E^h_t\}_{t\geq 0}$ to $\{E_t\}_{t\geq 0}$.

Theorem 6.2. *For any compact set* E_0 *in* $\mathbf{R}^N$, *let* $\{E^h_t\}_{t\geq 0}$ *and* $\{E_t\}_{t\geq 0}$ *be defined as above. Assume that* $[u(t,\cdot) \geq 0] = \overline{[u(t,\cdot) > 0]}$ *for each* $t \in [0,T)$. *Then* E^h_t *converges to* E_t *in the sense of Hausdorff distance as* $h \to 0$. *This convergence is locally uniform in* $[0,T)$.

Remark 6.1. Chambolle and Novaga considered in [7] and [8] approximation schemes to (1), which are the anisotropic versions of [3] and [6]. In these papers they obtained the Hausdorff convergence of the approximate flows. However, their convergence is only in the pointwise sense with respect to the t-variable and the result of Theorem 6.2 is sharper than theirs. Moreover, their proofs of the convergence consist of the construction of suitable sub- and super-solutions, based on the Gauss error function, and some variational techniques. These methods are different from the proofs of Theorems 6.1 and 6.2.

Acknowledgements. The research of Y. Giga was partly supported by the Grant-in-Aid of Scientific Research No. 21224001, the Japan Society for the Promotion of Science (JSPS). The research of K. Ishii was partly supported by the Grant-in-Aid of Scientific Research No. 21334028, JSPS.

References

[1] F. Almgren, J. E. Taylor and L. Wang, Curvature-driven flows: a variational approach, SIAM J. Control Optim. **31** (1993), no. 2, 387–438.

[2] G. Bellettini, V. Caselles, A. Chamboll and M. Novaga, Crystalline mean curvature flow of convex sets, Arch. Ration. Mech. Anal. **179** (2006), no. 1, 109–152.

[3] J. Bence, B. Merriman and S. Osher, Diffusion generated motion by mean curvature, in *Computational Crystal Growers Workshop*, J. Taylor ed. Selected Lectures in Math., Amer. Math. Soc., Province, 1992.

[4] F. Cao, *Geometric curve evolution and image processing*, Lecture Notes in Mathematics, 1805, Springer, Berlin, 2003.

[5] V. Caselles and A. Chambolle, Anisotropic curvature-driven flow of convex sets, Nonlinear Anal. **65** (2006), no. 8, 1547–1577.

[6] A. Chambolle, An algorithm for mean curvature motion, Interfaces Free Bound. **6** (2004), no. 2, 195–218.

[7] A. Chambolle and M. Novaga, Convergence of an algorithm for the anisotropic and crystalline mean curvature flow, SIAM J. Math. Anal. **37** (2006), no. 6, 1978–1987.

[8] A. Chambolle and M. Novaga, Approximation of the anisotropic mean curvature flow, Math. Models Methods Appl. Sci. **17** (2007), no. 6, 833–844.

[9] M. G. Crandall, H. Ishii and P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. (N.S.) **27** (1992), no. 1, 1–67.

[10] T. Eto, On a length minimizing scheme for the curve shortening problem, Master Thesis, University of Tokyo, 2008.

[11] Y. Giga, *Surface evolution equations*, Monographs in Mathematics, 99, Birkhäuser, Basel, 2006.

[12] Y. Giga, S. Goto, H. Ishii and M.-H. Sato, Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains, Indiana Univ. Math. J. **40** (1991), no. 2, 443–470.

[13] H. Ishii, G. E. Pires and P. E. Souganidis, Threshold dynamics type approximation schemes for propagating fronts, J. Math. Soc. Japan **51** (1999), no. 2, 267–308.

[14] G. Matheron, *Random sets and integral geometry*, John Wiley & Sons, New York, 1975.

[15] J. S. Moll, The anisotropic total variation flow, Math. Ann. **332** (2005), no. 1, 177–218.