Milnor K-groups modulo p^n of a complete discrete valuation field

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Abstract: For a mixed characteristic complete discrete valuation field K which contains a p^n-th root of unity, we determine the graded quotients of the filtration on the Milnor K-groups $K_q^M(K)$ modulo p^n in terms of differential forms of the residue field of K.

Key words: Milnor K-groups; complete discrete valuation field.

In higher dimensional local class field theory of K. Kato ([5,6] and [7]) the Galois group of an abelian extension field on a q-dimensional local field K is described by the Milnor K-group $K_q^M(K)$ for $q \geq 1^{*1)}$. The information on the ramification is related to the natural filtration $U^m K_q := U^m K_q^M(K)$ which is by definition the subgroup generated by $\{1 + \mathfrak{m}_K^m, K^\times, \dots, K^\times\}$, where $\mathfrak{m}_K$ is the maximal ideal of the ring of integers $\mathcal{O}_K$. So it is important to know the structure of the graded quotients $\mathrm{gr}^m K_q := U^m K_q / U^{m+1} K_q$. In this short note, we study the filtration on $k_{q,n} := K_q^M(K)/p^n K_q^M(K)$ the Milnor K-group modulo p^n induced by the filtration $U^m K_q$. More precisely, for a mixed characteristic complete discrete valuation field K, define the filtration $U^m k_{q,n}$ on $k_{q,n}$, by the image of the filtration $U^m K_q$ on $k_{q,n}$. Our objective is to determine the structure of its graded quotient $\mathrm{gr}^m k_{q,n} := U^m k_{q,n}/U^{m+1} k_{q,n}$ in terms of differential forms of the residue field of K under the assumption that K contains a primitive p^n-th root of unity ζ_{p^n} (Thm. 2).

It should be mentioned that J. Nakamura described $\mathrm{gr}^m k_{q,n}$ after determining $\mathrm{gr}^m K_q$ for all m when K is absolutely tamely ramified ([10], Cor. 1.2). Although it is easy in the case of $q = 1$, the structure of $\mathrm{gr}^m K_q$ is still unknown in general. In particular, when K has mixed characteristic and (absolutely) wildly ramification, it is known only some special cases ([9], see also [11]). As mentioned in [1], Remark 6.8, such structure is closely related to the number of roots of unity of p-primary orders in K. In fact, Kurihara treated a wildly ramified field with $\zeta_p \notin K$ in [9]. However, under the assumption $\zeta_{p^n} \in K$, the structure of $\mathrm{gr}^m k_{q,n}$ can be described by $\mathrm{gr}^m K_q$ only for lower m which is known by Bloch-Kato [1].

Let K be a complete discrete valuation field of characteristic 0, and k its residue field of characteristic $p > 0$. Let $e = v_K(p)$ be the absolute ramification index of K and $e_0 := e/(p-1)$. For $m \geq 1$, let $U^m K_q$ be the subgroup of $K_q^M(K)$ defined as above. Put $U^0 K_q = K_q^M(K)$ and $\mathrm{gr}^m K_q := U^m K_q / U^{m+1} K_q$. Let $\Omega_k^1 := \Omega_{k/\mathbf{Z}}^1$ be the module of absolute Kähler differentials and Ω_k^q the q-th exterior power of Ω_k^1 over the residue field k. Define the subgroups B_i^q and Z_i^q for $i \geq 0$ of Ω_k^q such that

$$0 = B_0^q \subset B_1^q \subset \cdots \subset Z_1^q \subset Z_0^q = \Omega_k^q$$

by the relations $B_1^q := \mathrm{Im}(d : \Omega_k^{q-1} \to \Omega_k^q)$, $Z_1^q := \mathrm{Ker}(d : \Omega_k^q \to \Omega_k^{q+1})$, $C^{-1} : B_i^q \xrightarrow{\simeq} B_{i+1}^q / B_1^q$, and $C^{-1} : Z_i^q \xrightarrow{\simeq} Z_{i+1}^q / B_1^q$, where $C^{-1} : \Omega_k^q \xrightarrow{\simeq} Z_1^q / B_1^q$ is the inverse Cartier operator defined by

$$(1) \qquad x \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_q}{y_q} \mapsto x^p \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_q}{y_q}.$$

We fix a prime element π of K. For any m, we have a surjective homomorphism $\rho_m : \Omega_k^{q-1} \oplus \Omega_k^{q-2} \to \mathrm{gr}^m K_q$ defined by

$$\left(x \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_{q-1}}{y_{q-1}}, 0 \right) \mapsto \{1 + \pi^m \widetilde{x}, \widetilde{y}_1, \dots \widetilde{y}_{q-1}\},$$

$$\left(0, x \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_{q-2}}{y_{q-2}} \right) \mapsto \{1 + \pi^m \widetilde{x}, \widetilde{y}_1, \dots \widetilde{y}_{q-2}, \pi\},$$

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*1) There is another approach to higher dimensional local class field theory developed by A. N. Parshin and I. B. Fesenko ([12], [2] and [3]). They adopted the topological Milnor K-group a quotient of the ordinal Milnor K-group.

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where $\widetilde{x}$ and $\widetilde{y}_i$ are liftings of x and y_i. For any $m \le e + e_0$, the kernel of ρ_m is written in terms of differential forms of k. Hence we obtain the structure of the graded quotient $\mathrm{gr}^m K_q$ ([1], see also [11]) and also $\mathrm{gr}^m k_{q,n}$ ([1], Rem. 4.8). Recall that the filtration $U^m k_{q,n}$ on $k_{q,n} = K_q^M(K)/p^n K_q^M(K)$ is defined by the image of the filtration $U^m K_q$ on $k_{q,n}$ and $\mathrm{gr}^m k_{q,n} := U^m k_{q,n}/U^{m+1}k_{q,n}$. From the following lemma, we can investigate $\mathrm{gr}^m k_{q,n}$ for $m > e + e_0$ by its structure for $m \le e + e_0$.

Lemma 1. *For $n > 1$ and $m > e + e_0$, the multiplication by p induces a surjective homomorphism $p : U^{m-e}k_{q,n-1} \to U^m k_{q,n}$. If we further assume $\zeta_{p^n} \in K$, then the map p is bijective.*

Proof. The surjectivity of $p : U^{m-e}k_{q,n-1} \to U^m k_{q,n}$ follows from the surjectivity of $p : U^{m-e}k_{1,n-1} \to U^m k_{1,n}$. To show the injectivity, for $x \in U^{m-e}K_q$, assume that $px = p^n x'$ is in $p^n K_q^M(K) \cap U^m K_q$ for some $x' \in K_q^M(K)$. Thus $x - p^{n-1}x'$ is in the kernel of the multiplication by p on $K_q^M(K)$. It is known that its kernel is $= \{\zeta_p\}K_{q-1}^M(K)$, where ζ_p is a primitive p-th root of unity. This fact is a byproduct of the Milnor-Bloch-Kato conjecture (due to Suslin, *cf.* [8], Sect. 2.4), now is a theorem of Voevodsky, Rost, and Weibel ([13]). Hence, for any $y \in K_{q-1}^M(K)$, we have $\{\zeta_p, y\} = p^{n-1}\{\zeta_{p^n}, y\}$ and thus $x \in p^{n-1}K_q^M(K)$. □

We determine $\mathrm{gr}^m k_{q,n}$ for any m and n when ζ_{p^n} is in K. It is known also $U^m k_{q,1} = 0$ for $m > e + e_0$ ([1], Lem. 5.1 (i)). So we may assume $m > e + e_0$ and $n > 1$. For such m, we have an isomorphism $\mathrm{gr}^{m-e} k_{q,n-1} \xrightarrow{p} \mathrm{gr}^m k_{q,n}$ from the above lemma. By induction on n, we obtain the following

Theorem 2. *We assume $\zeta_{p^n} \in K$. Let m and n be positive integers and s the integer such that $m = p^s m'$, $(m', p) = 1$. Put $c_i := ie + e_0$ for $i \ge 1$ and $c_0 := 0$.*

(i) *If $c_i < m < c_{i+1}$ for some $0 \le i < n$, then $\mathrm{gr}^m k_{q,n}$ is isomorphic to*

$$\begin{cases} \mathrm{Coker}(\Omega_k^{q-2} \xrightarrow{\theta} \Omega_k^{q-1}/B_s^{q-1} \oplus \Omega_k^{q-2}/B_s^{q-2}), n - i > s, \\ \Omega_k^{q-1}/Z_{n-i}^{q-1} \oplus \Omega_k^{q-2}/Z_{n-i}^{q-2},\ n - i \le s, \end{cases}$$

where θ is defined by

$$\omega \mapsto (C^{-s}d\omega, (-1)^q(m - ie)/p^s C^{-s}\omega).$$

(ii) *If $m = c_i$ for some $0 < i \le n$, then $\mathrm{gr}^{ie+e_0} k_{q,n}$ is isomorphic to*

$$(\Omega_k^{q-1}/(1 + aC)Z_{n-i}^{q-1}) \oplus (\Omega_k^{q-2}/(1 + aC)Z_{n-i}^{q-2}),$$

where C is the Cartier operator defined by

$$x^p \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_{q-1}}{y_{q-1}} \mapsto x \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_{q-1}}{y_{q-1}}$$

and a is the residue class of $p\pi^{-e}$.

(iii) *If $m > c_n$, then $U^m k_{q,n} = 0$.*

Corollary 3. *If k is separably closed (may not assume $\zeta_{p^n} \in K$), then $\mathrm{gr}^{ie+e_0} k_{q,n} = 0$ for $i \ge 1$.*

Proof. The assertion follows from $\mathrm{gr}^{e+e_0} k_{q,1} = 0$ ([1], Lem. 5.1 (ii)), Lemma 1, and the induction on n. □

We conclude this note to give a remark: If we further assume K has a structure of a higher dimensional local field, we have $K_q^M(K)/p^n K_q^M(K) \simeq K_q^{\mathrm{top}}(K)/p^n K_q^{\mathrm{top}}(K)$, where $K_q^{\mathrm{top}}(K)$ is the topological Milnor K-group. The structure of the later group has been fully studied by using the Vostokov symbol in [2] and [4].

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