On mean ergodic semigroups of random linear operators

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Abstract: In this paper, we prove a mean ergodic theorem for an almost surely bounded strongly continuous semigroup of random linear operators on a random reflexive random normed module, which generalizes and improves several known important results.

Key words: Random normed module; ergodic semigroup of random linear operators; random reflexivity.

1. Introduction. The notion of a random normed module (briefly, an *RN* module), which was first introduced in [6] and subsequently elaborated in [7], is a random generalization of that of a normed space. Since an *RN* module is often endowed with a natural topology, called the (ε, λ)–topology, it is not a locally convex space in general and in particular the theory of classical conjugate spaces universally fails to serve the theory of *RN* modules. The theory of random conjugate spaces for *RN* modules has been developed in order to overcome the obstacle [6,7]. Subsequently, the theory of *RN* modules together with their random conjugate spaces has obtained a systematic and deep development in the direction of functional analysis [3–5,8–12], in particular the random reflexivity based on the theory of random conjugate spaces and the study of semigroups of random linear operators have also obtained some substantial advances in [8,10,12,14,18]. The purpose of this paper is to further study the mean ergodicity of semigroups of random linear operators on a random reflexive *RN* module.

Motivated by the works of Muštari and Taylor [15,17], we have recently begun to study the mean ergodic theorem under the framework of *RN* modules [18] to obtain the mean ergodic theorem in the sense of convergence in probability, where we proved the mean ergodic theorem for a strongly continuous semigroup of random unitary operators defined on complete random inner product modules (briefly, complete *RIP* modules). It is clear that a strongly continuous semigroup of random unitary operators is almost surely bounded and a complete *RIP* module is random reflexive. Since every bounded strongly continuous semigroup of linear operators on a reflexive Banach space is mean ergodic [2], this motivates us, in this paper, to further prove a mean ergodic theorem for an almost surely bounded strongly continuous semigroup of random linear operators on a random reflexive random normed module, so that results in this paper considerably generalize and improve those in [18]. It should be pointed out the connection between random reflexivity of a complete *RN* module $\tilde{I}_A S$ and the reflexivity of the Banach space $L^p(\tilde{I}_A S)$ $(1 < p < \infty)$ will play a crucial role in this paper, where $\tilde{I}_A S$ denotes the A-stratification of S (see [11] or see Section 3 for the notion of A-stratification) and $L^p(\tilde{I}_A S)$ the Banach space generated by $\tilde{I}_A S$.

The remainder of this paper is organized as follows: In Section 2 we briefly recall some necessary basic notions and facts; in Section 3 we present our main result and its proof.

2. Preliminaries. Throughout this paper, $(\Omega, \mathcal{F}, P)$ denotes a probability space, K the scalar field R of real numbers or C of complex numbers, $\bar{L}^0(\mathcal{F}, R)$ the set of equivalence classes of extended real-valued $\mathcal{F}$-measurable random variables on Ω, $L^0(\mathcal{F}, K)$ the algebra of equivalence classes of K-valued $\mathcal{F}$-measurable random variables on Ω under the ordinary addition, scalar multiplication and multiplication operations on equivalence classes.

It is well known from [1] that $\bar{L}^0(\mathcal{F}, R)$ is a complete lattice under the ordering $\leqslant$: $\xi \leqslant \eta$ if and only if $\xi^0(\omega) \leq \eta^0(\omega)$ for P-almost all ω in

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Ω (briefly, a.s.), where ξ^0 and η^0 are arbitrarily chosen representatives of ξ and η, respectively. Furthermore, every subset A of $\bar{L}^0(\mathcal{F}, R)$ has a supremum, denoted by $\vee A$, and an infimum, denoted by $\wedge A$, and there exist two sequences $\{a_n, n \in N\}$ and $\{b_n, n \in N\}$ in A such that $\vee_{n\geqslant 1} a_n = \vee A$ and $\wedge_{n\geqslant 1} b_n = \wedge A$. Finally, $L^0(\mathcal{F}, R)$, as a sublattice of $\bar{L}^0(\mathcal{F}, R)$, is complete in the sense that every subset with an upper bound has a supremum.

As usual, we denote $L^0_+ = \{\xi \in L^0(\mathcal{F}, R) \mid \xi \geqslant 0\}$.

Definition 2.1[7,3]. An ordered pair $(S, \|\cdot\|)$ is called an RN module over K with base $(\Omega, \mathcal{F}, P)$ if S is a left module over the algebra $L^0(\mathcal{F}, K)$ and $\|\cdot\|$ is a mapping from S to L^0_+ such that the following three axioms are satisfied:

(1) $\|\xi x\| = |\xi|\|x\|, \forall \xi \in L^0(\mathcal{F}, K)$ and $x \in S$;

(2) $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in S$;

(3) $\|x\| = 0$ implies $x = \theta$ (the null vector of S), where $\|\cdot\|$ is called the L^0–norm on S and $\|x\|$ is called the L^0–norm of a vector $x \in S$.

It should be pointed out that the following idea of introducing the (ε, λ)–topology is due to B. Schweizer and A. Sklar [16].

Let $(S, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$. For any positive real numbers ε and λ such that $\lambda < 1$, let $N_\theta(\varepsilon, \lambda) = \{x \in S \mid P\{\omega \in \Omega \mid \|x\|(\omega) < \varepsilon\} > \lambda\}$, then $\{N_\theta(\varepsilon, \lambda) \mid \varepsilon > 0, 0 < \lambda < 1\}$ is a local base at the null vector θ of some Hausdorff linear topology. The linear topology is called the (ε, λ)–topology. In this paper, given an RN module $(S, \|\cdot\|)$ over K with base $(\Omega, \mathcal{F}, P)$, it is always assumed that $(S, \|\cdot\|)$ is endowed with the (ε, λ)–topology. One only needs to notice that a sequence $\{x_n, n \in N\}$ in S converges to $x \in S$ in the (ε, λ)–topology if and only if $\{\|x_n - x\|, n \in N\}$ converges to 0 in probability P.

Example 2.1. Define the mapping $\|\cdot\| : L^0(\mathcal{F}, K) \to L^0_+$ by $\|x\| = |x|$ for any $x \in L^0(\mathcal{F}, K)$, where $|x|$ is the equivalence class of the composite function $|x^0| : \Omega \to [0, +\infty)$ defined by $|x^0|(\omega) = |x^0(\omega)|$ for any $\omega \in \Omega$, while x^0 is an arbitrary chosen representative of x. Then $\|\cdot\|$ is an L^0–norm on $L^0(\mathcal{F}, K)$ and $(L^0(\mathcal{F}, K), \|\cdot\|)$ is an RN module over K with base $(\Omega, \mathcal{F}, P)$.

Definition 2.2[14,13]. Let $(S^1, \|\cdot\|_1)$ and $(S^2, \|\cdot\|_2)$ be two RN modules over K with base $(\Omega, \mathcal{F}, P)$. A linear operator T from S^1 to S^2 is called a random linear operator, further, the random linear operator T is called a.s. bounded if there exists a $\xi \in L^0_+$ such that $\|Tx\|_2 \leq \xi \cdot \|x\|_1$ for any $x \in S^1$. Denote by $B(S^1, S^2)$ the linear space of a.s. bounded random linear operators from S^1 to S^2, define $\|\cdot\| : B(S^1, S^2) \to L^0_+$ by $\|T\| := \wedge\{\xi \in L^0_+ \mid \|Tx\|_2 \leq \xi \cdot \|x\|_1, \ \forall x \in S^1\}$ for any $T \in B(S^1, S^2)$, then it is easy to see that $(B(S^1, S^2), \|\cdot\|)$ is an RN module over K with base $(\Omega, \mathcal{F}, P)$.

Specially, denote $(B(S^1, S^2), \|\cdot\|)$ by $(S^*, \|\cdot\|^*)$ when $(S^1, \|\cdot\|_1)$ is a given RN module $(S, \|\cdot\|)$ over K with base $(\Omega, \mathcal{F}, P)$ and $S^2 = L^0(\mathcal{F}, K)$, then $(S^*, \|\cdot\|^*)$ is called the random conjugate space of $(S, \|\cdot\|)$. Let $(S^{**}, \|\cdot\|^{**})$ be the random conjugate space of $(S^*, \|\cdot\|^*)$. The canonical embedding mapping $J : S \to S^{**}$ defined by $(Jx)(f) = f(x)$ for any $x \in S$ and $f \in S^*$, is random–norm preserving. If J is surjective, then S is called random reflexive [12].

The Definition 2.3 below is essentially from [10].

Definition 2.3. Let $(S, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$, $B(S)$ the set of a.s. bounded random linear operators on S. A family $\{B(t) : t \geq 0\} \subset B(S)$ is called a semigroup of random linear operators if

$$B(0) = I \text{ and } B(s)B(t) = B(s + t)$$

for all $s, t \geq 0$, where I denotes the identity operator on S. Further, if the mapping $B(\cdot)x : [0, +\infty) \to S$, $t \mapsto B(t)x$ is continuous w.r.t. the (ε, λ)–topology for every $x \in S$, then the semigroup of random linear operators $\{B(t) : t \geq 0\}$ is said to be strongly continuous. Besides, if $\vee_{t\geq 0}\|B(t)\| \in L^0_+$, then $\{B(t) : t \geq 0\}$ is called an a.s. bounded strongly continuous semigroup of random linear operators.

Proposition 2.1[14,13]. *Let $(S^1, \|\cdot\|_1)$ and $(S^2, \|\cdot\|_2)$ be two RN modules over K with base $(\Omega, \mathcal{F}, P)$. Then we have the following statements*:

(1) *$T \in B(S^1, S^2)$ if and only if T is a continuous module homomorphism*;

(2) *If $T \in B(S^1, S^2)$, then $\|T\| = \vee\{\|Tx\|_2 : x \in S^1$ and $\|x\|_1 \leq 1\}$, where 1 denotes the identity element in $L^0(\mathcal{F}, R)$.*

3. The main result and its proof. The main result of this paper is Theorem 3.1 below, whose proof needs Lemma 3.1 as well as Propositions 3.2 and 3.3. For the reader's convenience, let us first recall the definition of Riemann integral for abstract-valued functions from a finite real interval to an RN module and a sufficient condition for such a function to be Riemann-integrable [10].

Let $[a,b]$ be a finite real interval and $\mathcal{P}=\{x_1,x_2,\ldots,x_k,\ldots,x_n\}$ a finite partition into $[a,b]$, namely, $a=x_1<x_2<\ldots<x_{n-1}<x_n=b$ and $\lambda(\mathcal{P})=\max\limits_{1\le i\le n}\{\Delta x_i\}$, where $\Delta x_i=x_i-x_{i-1}$ $(i=1,\ldots,n)$. Besides, from now on we always suppose that $(S,\|\cdot\|)$ denotes a complete RN module over K with base $(\Omega,\mathcal{F},P)$.

Definition 3.1[10]**.** Let f be a function from $[a,b]$ to S. f is called Riemann integrable on $[a,b]$ if there exists some I in S with the following property: for any positive numbers ε and λ with $\lambda<1$ there is a positive number $\delta(\varepsilon,\lambda)$ such that

$$P\left\{\omega\in\Omega \;\middle|\; \left\|\sum_{i=1}^{n}f(\xi_i)\Delta x_i-I\right\|(\omega)<\varepsilon\right\}>1-\lambda$$

for any finite partition $\mathcal{P}=\{x_1,x_2,\ldots,x_k,\ldots,x_n\}$ and arbitrarily chosen $\xi_i\in[x_{i-1},x_i]$ $(1\le i\le n)$ whenever $\lambda(\mathcal{P})<\delta(\varepsilon,\lambda)$. Further I is called the Riemann integral of f in the (ε,λ)–topology over $[a,b]$, denoted by $\int_a^b f(t)dt$.

Proposition 3.1[10]**.** *Let f be a continuous function from $[a,b]$ to S such that $\vee_{t\in[a,b]}\|f(t)\|\in L^0_+$, then f is Riemann integrable in the (ε,λ)–topology on $[a,b]$.*

We can now introduce Definition 3.2 below.

Definition 3.2. Let $\{B(t):t\ge 0\}$ be an a.s. bounded strongly continuous semigroup of random linear operators on an RN module S. We denote by

$$(1)\qquad C(r)x:=\frac{1}{r}\int_0^r B(s)xds,\quad \forall x\in S,\ r>0$$

the Cesàro means of $\{B(t):t\ge 0\}$. For any $x\in S$, if $\{C(r)x,\ r>0\}$ converges to some point in S as $r\to\infty$, then $\{B(t):t\ge 0\}$ is called mean ergodic.

Remark 3.1. In Definition 3.2, for any fixed $x\in S$ and $r>0$, Let $f(t)=B(t)x$ for any $t\in[0,r]$, then f is an abstract function from $[0,r]$ to S and $\vee_{t\in[0,r]}\|f(t)\|\in L^0_+$, thus f is Riemann integrable in the (ε,λ)–topology on $[0,r]$ by Proposition 3.1, which shows that the equation (1) is well defined.

Based on the above preliminaries, we can now state Theorem 3.1 as follows:

Theorem 3.1. *Let $(S,\|\cdot\|)$ be a complete RN module over K with base $(\Omega,\mathcal{F},P)$. If S is random reflexive, then every a.s. bounded strongly continuous semigroup of random linear operators on S is mean ergodic.*

In this paper we distinguish random variables from their equivalence classes by means of symbols: for example, I_A denotes the characteristic function of the $\mathcal{F}$–measurable set A, then we use $\tilde{I}_A$ for its equivalence class. Besides, let $A=\{\omega\in\Omega\mid \xi^0(\omega)>\eta^0(\omega)\}$, where ξ^0 and η^0 are arbitrarily chosen representatives of ξ and η in $L^0(\mathcal{F},R)$, respectively, then we always use $[\xi>\eta]$ for the equivalence class of A and often write $I_{[\xi>\eta]}$ for $\tilde{I}_A$, one can also understand such notations as $I_{[\xi\le\eta]}$, $I_{[\xi\ne\eta]}$ and $I_{[\xi=\eta]}$.

In the sequel of this section, Let $(S,\|\cdot\|)$ be a complete RN module over K with base $(\Omega,\mathcal{F},P)$, p a given positive number such that $1<p<\infty$ and $L^p(S)=\{x\in S\mid [\int_\Omega\|x\|^pdP]^{\frac{1}{p}}<+\infty\}$. Define the mapping $\|\cdot\|_p:L^p(S)\to[0,+\infty)$ by $\|x\|_p=[\int_\Omega\|x\|^pdP]^{\frac{1}{p}}$ for any $x\in L^p(S)$, then $(L^p(S),\|\cdot\|_p)$ is an ordinary Banach space. Let $B(L^p(S))$ denote the set of bounded linear operators on $L^p(S)$. Obviously, for an $\mathcal{F}$-measurable subset A of Ω and an $L^0(\mathcal{F},K)$-module S, $\tilde{I}_AS:=\{\tilde{I}_Ax\mid x\in S\}$, called the A-stratification of S, is a left module over the algebra $\tilde{I}_AL^0(\mathcal{F},K):=\{\tilde{I}_A\xi\mid \xi\in L^0(\mathcal{F},K)\}$ and $(L^p(\tilde{I}_AS),\ \|\cdot\|_p)$ is an ordinary Banach space.

Now let $\{B(t):t\ge 0\}$ be an a.s. bounded strongly continuous semigroup of random linear operators on S and $\xi=\vee_{t\ge 0}\|B(t)\|$, then $\xi\in L^0_+$. Let $E_i=[i-1\le\xi<i]$ for each $i\in N$, then $\{E_i,\ i\ge 1\}$ is a sequence of pairwise disjoint $\mathcal{F}$–measurable sets such that $\sum_{i=1}^{\infty}E_i=\Omega$. Further, if we take $B_i(t)=I_{E_i}\cdot B(t)$ for any $i\in N$ and $t\ge 0$, then Lemma 3.1 *below holds.*

Lemma 3.1. *$\{B_i(t):t\ge 0\}$ is a bounded strongly continuous semigroup of linear operators on the Banach space $L^p(I_{E_i}S)$.*

Proof. Since

$$(2)\qquad \begin{aligned}\|B_i(t)x\|_p&=\left[\int_\Omega\|B_i(t)x\|^pdP\right]^{\frac{1}{p}}\\ &\le\left[\int_\Omega\|B_i(t)\|^p\cdot\|x\|^pdP\right]^{\frac{1}{p}}\\ &\le i\cdot\left[\int_\Omega\|x\|^pdP\right]^{\frac{1}{p}}\\ &=i\cdot\|x\|_p\end{aligned}$$

for any $x\in L^p(I_{E_i}S)$, $i\in N$ and $t\ge 0$, it follows that the restriction of $B_i(t)$ to $L^p(I_{E_i}S)$, namely $B_i(t)|_{L^p(I_{E_i}S)}$ (still denoted by $B_i(t)$) $\in B(L^p(I_{E_i}S))$ and $|||B_i(t)|||\le i$, where $|||B_i(t)|||$ denotes the ordinary operator norm of $B_i(t)$. Thus it is easy to see that $\sup_{t\geq 0}|||B_i(t)|||<\infty$ for each fixed $i\in N$. Furthermore,

$$B_i(0)x = B(0)(I_{E_i}x) = I_{E_i}x = x$$

for any $x\in L^p(I_{E_i}S)$ and

$$B_i(t)B_i(s) = I_{E_i}B(t+s) = B_i(t+s)$$

for any $t,s\geq 0$. Since

$$\|B_i(t)x - B_i(0)x\|^p \leq (2i)^p\cdot\|x\|^p \in L^1(I_{E_i}S)$$

for any $x\in L^p(I_{E_i}S)$ and it is easy to see that $\|B_i(t)x-B_i(0)x\|^p\to 0$ $(t\to 0)$ in probability P, it follows from Lebesgue's dominated convergence theorem that

$$\begin{aligned}(3)\qquad &\|B_i(t)x - B_i(0)x\|_p\\ &= \left(\int_\Omega \|B_i(t)x-B_i(0)x\|^p dP\right)^{\frac{1}{p}}\\ &\to 0,\ \text{as } t\to 0.\end{aligned}$$

Consequently, $\{B_i(t): t\geq 0\}$ is a bounded strongly continuous semigroup of linear operators on the Banach space $L^p(I_{E_i}S)$ for each fixed $i\in N$. □

Proposition 3.2[12]. *An RN module S is random reflexive if and only if $L^p(S)$ is reflexive for any given p such that $1<p<\infty$.*

Proposition 3.3[2]. *Let X be a Banach space. If X is reflexive, then every bounded strongly continuous semigroup of linear operators on X is mean ergodic.*

We can now prove Theorem 3.1.

Proof of Theorem 3.1. For each $i\in N$ and $x\in S$, let $x^i = I_{E_i}x$, then $x^i\in I_{E_i}S$. Let $F_j^i=[j-1\leq\|x^i\|<j]$ for each $j\in N$, then $\{F_j^i, j\geq 1\}$ is a sequence of pairwise disjoint $\mathcal{F}$–measurable sets such that $\sum_{j=1}^\infty F_j^i=\Omega$. Let $x_j^i = I_{F_j^i}\cdot x^i$ for each $j\in N$, then $x_j^i\in L^p(I_{E_i}S)$ for each $j\in N$. Since $(S,\|\cdot\|)$ is random reflexive, it is clear that $I_{E_i}S$ is random reflexive. Thus it follows from Proposition 3.2 that $L^p(I_{E_i}S)$ is a reflexive Banach space. Further, by Lemma 3.1, we have $\{B_i(t): t\geq 0\}$ is a bounded strongly continuous semigroup of linear operators on $L^p(I_{E_i}S)$. Consequently, for each fixed $i\in N$, $\frac{1}{r}\int_0^r B_i(s)x_j^i ds$ is mean ergodic on $L^p(I_{E_i}S)$ for each $r>0$ by Proposition 3.3, namely, for each $j\in N$, there exists a $y_j^i\in L^p(I_{E_i}S)$ such that $\frac{1}{r}\int_0^r B_i(s)x_j^i ds$ converges to y_j^i in $\|\cdot\|_p$ as $r\to\infty$, and hence $\frac{1}{r}\int_0^r B_i(s)x_j^i ds$ also converges to y_j^i in the (ε,λ)–topology as $r\to\infty$. Observe that

$$y_j^i = I_{F_j^i}\cdot y_j^i \quad\text{and}\quad x_j^i = I_{F_j^i}\cdot x_j^i$$

for each $j\in N$ and we can suppose $P(F_j^i)>0$ for each $j\in N$ (otherwise such an F_j^i is automatically omitted). Moreover, since

$$\sum_{j=1}^\infty P(F_j^i) = P\left(\sum_{j=1}^\infty F_j^i\right) = P(\Omega) = 1,$$

it follows that $\{\sum_{j=1}^m y_j^i, m\in N\}$ is a Cauchy sequence in S. Since S is complete, it follows that there exists some y^i in S such that $\{\sum_{j=1}^m y_j^i, m\in N\}$ converges to y^i as $m\to\infty$, namely, for any $\varepsilon,\delta>0$ such that $\delta<1$, there exists an $M(\varepsilon,\delta)\in N$ such that $m\geq M(\varepsilon,\delta)$ implies that $P[\|\sum_{j=1}^m y_j^i - y^i\|\geq\frac{\varepsilon}{3}]<\frac{\delta}{3}$. Choose an $m_1\geq M(\varepsilon,\delta)$ such that $P[\|\sum_{j=1}^{m_1} y_j^i - y^i\|\geq\frac{\varepsilon}{3}]<\frac{\delta}{3}$. Obviously, $\{\sum_{j=1}^m\frac{1}{r}\int_0^r B_i(s)x_j^i ds, m\in N\}$ converges to $\frac{1}{r}\int_0^r B_i(s)x^i ds$ as $m\to\infty$ for each $r>0$. Similarly, for the same $\varepsilon,\delta>0$, we can choose an $m_2\in N$ such that $P[\|\sum_{j=1}^{m_2}\frac{1}{r}\int_0^r B_i(s)x_j^i ds - \frac{1}{r}\int_0^r B_i(s)x^i ds\|\geq\frac{\varepsilon}{3}]<\frac{\delta}{3}$ for each $r>0$. Let $m_0=m_1\vee m_2$, we have

$$(4)\qquad P\left[\left\|\sum_{j=1}^{m_0} y_j^i - y^i\right\|\geq\frac{\varepsilon}{3}\right]<\frac{\delta}{3}$$

and

$$(5)\quad P\left[\left\|\sum_{j=1}^{m_0}\frac{1}{r}\int_0^r B_i(s)x_j^i ds - \frac{1}{r}\int_0^r B_i(s)x_i ds\right\|\geq\frac{\varepsilon}{3}\right] <\frac{\delta}{3}$$

for each $r>0$. It is easy to see that $\sum_{j=1}^{m_0}\frac{1}{r}\int_0^r B_i(s)x_j^i ds$ converges to $\sum_{j=1}^{m_0} y_j^i$ in the (ε,λ)–topology as $r\to\infty$, namely, for the same $\varepsilon,\delta>0$ above, there exists an $N(\varepsilon,\delta)\in N$ such that $r\geq N(\varepsilon,\delta)$ implies that

$$(6)\quad P\left[\left\|\sum_{j=1}^{m_0}\frac{1}{r}\int_0^r B_i(s)x_j^i ds - \sum_{j=1}^{m_0} y_j^i\right\|\geq\frac{\varepsilon}{3}\right]<\frac{\delta}{3}.$$

Since

$$\begin{aligned}(7)\quad &\left\|\frac{1}{r}\int_0^r B_i(s)x^i ds - y^i\right\|\\ &\leq \left\|\frac{1}{r}\int_0^r B_i(s)x^i ds - \sum_{j=1}^{m_0}\frac{1}{r}\int_0^r B_i(s)x_j^i ds\right\|\\ &+\left\|\sum_{j=1}^{m_0}\frac{1}{r}\int_0^r B_i(s)x_j^i ds - \sum_{j=1}^{m_0} y_j^i\right\|\\ &+\left\|\sum_{j=1}^{m_0} y_j^i - y^i\right\|\end{aligned}$$

for each $r>0$, it follows that

$$(8)\quad \left[\left\|\frac{1}{r}\int_0^r B_i(s)x^i ds - y^i\right\| \geq \varepsilon\right]$$
$$\subset \left[\left\|\frac{1}{r}\int_0^r B_i(s)x^i ds - \sum_{j=1}^{m_0}\frac{1}{r}\int_0^r B_i(s)x^i_j ds\right\| \geq \frac{\varepsilon}{3}\right]$$
$$\cup \left[\left\|\sum_{j=1}^{m_0}\frac{1}{r}\int_0^r B_i(s)x^i_j ds - \sum_{j=1}^{m_0} y^i_i\right\| \geq \frac{\varepsilon}{3}\right]$$
$$\cup \left[\left\|\sum_{j=1}^{m_0} y^i_j - y^i\right\| \geq \frac{\varepsilon}{3}\right]$$

for each $r > 0$. Thus, according to (4), (5) and (6), (8) gives

$$(9)\quad P\left[\left\|\frac{1}{r}\int_0^r B_i(s)x^i ds - y^i\right\| \geq \varepsilon\right]$$
$$\leq P\left[\left\|\frac{1}{r}\int_0^r B_i(s)x^i ds - \sum_{j=1}^{m_0}\frac{1}{r}\int_0^r B_i(s)x^i_j ds\right\| \geq \frac{\varepsilon}{3}\right]$$
$$+ P\left[\left\|\sum_{j=1}^{m_0}\frac{1}{r}\int_0^r B_i(s)x^i_j ds - \sum_{j=1}^{m_0} y^i_j\right\| \geq \frac{\varepsilon}{3}\right]$$
$$+ P\left[\left\|\sum_{j=1}^{m_0} y^i_j - y^i\right\| \geq \frac{\varepsilon}{3}\right]$$
$$< \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta$$

as $r \geq N(\varepsilon, \delta)$, which implies that $\frac{1}{r}\int_0^r B_i(s)x^i ds$ converges to y^i as $r \to \infty$ in the (ε, λ)–topology for each fixed $i \in N$.

Since

$$y^i = I_{E_i} \cdot y^i \quad \text{and} \quad B_i(t)x^i = I_{E_i} \cdot B_i(t)x^i$$

for each $i \in N$, $t \geq 0$ and

$$\sum_{i=1}^{\infty} P(E_i) = P\left(\sum_{i=1}^{\infty} E_i\right) = P(\Omega) = 1,$$

let $y = \sum_{i=1}^{\infty} y^i$, similar to the above proof, one can obtain that $\frac{1}{r}\int_0^r B(s)x ds$ converges to y as $n \to \infty$ in the (ε, λ)–topology.

This completes the proof. □

One of the main results of [18] is Corollary 3.1 below, whose proof was long in [18], whereas based on Theorem 3.1 we can give it a concise proof.

Corollary 3.1[18]**.** *Let S be a complete random inner product module over C with base $(\Omega, \mathcal{F}, P)$, $\{U(t) : t \geq 0\}$ a strongly continuous semigroup of random unitary operators on S and P_0 the random orthogonal projection onto the submodule $S_0 = \{x \in S \mid U(t)x = x, \ \forall t \geq 0\}$. Then*

$$\lim_{r\to\infty} \frac{1}{r}\int_0^r U(t)x dt = P_0 x, \quad \forall x \in S.$$

Proof. Since every complete random inner product module is random reflexive, it follows from Theorem 3.1 that there exists some $y \in S$ such that $\frac{1}{r}\int_0^r U(t)x dt$ converges to y as $r \to \infty$ in the (ε, λ)–topology. Thus it remains to prove that $U(t)y = y$ for each $t \geq 0$.

Since

$$(10)\quad \frac{1}{r}\int_s^{r+s} U(t)x dt$$
$$= \frac{1}{r}\int_0^{r+s} U(t)x dt - \frac{1}{r}\int_0^s U(t)x dt$$
$$= \frac{r+s}{r}\frac{1}{r+s}\int_0^{r+s} U(t)x dt$$
$$- \frac{1}{r}\int_0^s U(t)x dt$$

for any r, $s > 0$ and $x \in S$, fix s and x, letting $r \to \infty$ in (10) yields that $\frac{1}{r}\int_s^{r+s} U(t)x dt$ converges to y. Observe that

$$(11)\quad U(s)\left(\frac{1}{r}\int_0^r U(t)x dt\right)$$
$$= \frac{1}{r}\int_0^r U(s+t)x dt$$
$$= \frac{1}{r}\int_s^{r+s} U(t)x dt,$$

and the desired result follows.

This complete the proof. □

Remark 3.2. Since Corollary 3.1 was proved by using Stone's representation theorem on the complete complex random inner product module S in [10], S must be required to be on C. Whereas if C is taken the place of K in Corollary 3.1, by Theorem 3.1, it still holds.

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