On the Mathieu mock theta function

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Abstract: In this paper, we study the congruences of the Fourier coefficients of the Mathieu mock theta function, which appears in the Mathieu moonshine phenomenon discovered by Eguchi, Ooguri, and Tachikawa.

Key words: Mock theta functions; Mathieu group; moonshine; Maass forms; modular forms.

1. Introduction. Let $E_4(\tau)$ be the Eisenstein series and $\eta(\tau)$ the Dedekind η function:

$$E_4(\tau) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n,$$

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n),$$

where $\sigma_3(n) = \sum_{m|n} m^3$. Then the j-function is defined as follows:

$$\begin{aligned} j(\tau) &= \frac{E_4(\tau)^3}{\eta(\tau)^{24}} = \frac{1}{q} + 744 + 196884q + \cdots \\ &= \sum_{n=-1}^{\infty} c(n)q^n \text{ (say).} \end{aligned}$$

McKay noticed that $c(1) = 196884 = 196883 + 1$ for the first nontrivial coefficient of the function $j(\tau)$ and the first nontrivial dimension of the irreducible representations of the Monster group. Based on these observations, Conway and Norton formulated the Moonshine conjecture, which has now been proved by Borcherds [1,2].

It is remarkable that the coefficients $c(n)$ have the following congruences [6] for $a \geq 1$:

$$\begin{aligned} n &\equiv 0 \pmod{2^a} &&\Rightarrow c(n) \equiv 0 \pmod{2^{3a+8}} \\ n &\equiv 0 \pmod{3^a} &&\Rightarrow c(n) \equiv 0 \pmod{3^{2a+3}} \\ n &\equiv 0 \pmod{5^a} &&\Rightarrow c(n) \equiv 0 \pmod{5^{a+1}} \\ n &\equiv 0 \pmod{7^a} &&\Rightarrow c(n) \equiv 0 \pmod{7^{a}} \\ n &\equiv 0 \pmod{11^a} &&\Rightarrow c(n) \equiv 0 \pmod{11^{a}}. \end{aligned}$$

In 2010, Eguchi, Ooguri, and Tachikawa discovered the Mathieu moonshine phenomenon [3]. Let $\theta_1(z;\tau)$ and $\mu(z;\tau)$ be the following functions:

$$\theta_1(z;\tau) = -\sum_{n=-\infty}^{\infty} e^{\pi i\tau(n+\frac{1}{2})^2 + 2\pi i(n+\frac{1}{2})(z+\frac{1}{2})},$$

$$\mu(z;\tau) = \frac{ie^{\pi iz}}{\theta_1(z;\tau)}\sum_{n\in\mathbf{Z}}(-1)^n\frac{q^{\frac{1}{2}n(n+1)}e^{2\pi inz}}{1-q^ne^{2\pi iz}}.$$

We set $\Sigma(\tau)$ as follows:

$$\begin{aligned} \Sigma(\tau) &:= 8\sum_{z\in\{1/2,\tau/2,(1+\tau)/2\}}\mu(z;\tau) \\ &= q^{-\frac{1}{8}}\left(2 - \sum_{n=1}^{\infty}A(n)q^n\right) \text{ (say)} \\ &= -q^{-\frac{1}{8}}(-2 + 90q + 462q^2 + 1540q^3 \\ &\quad + 4554q^4 + 11592q^5 + 27830q^6 + \cdots). \end{aligned}$$

Then the Mathieu moonshine phenomenon is that the first 5 coefficients divided by 2,

$$\{45, 231, 770, 2277, 5796\},$$

are equal to the dimensions of irreducible representations of M_{24} and other coefficients can be written as linear combinations of dimensions of the Mathieu group M_{24}. The reason for this mysterious phenomenon is still unknown.

Recently, the author found some congruences of the coefficients $A(n)$ of the Mathieu mock theta function and we conjectured as follows:

(1)

$$\begin{aligned} &n \equiv 1,2 \pmod 3 &&\Rightarrow A(n) \equiv 0 \pmod 3 \\ &n \equiv 1,3 \pmod 5 &&\Rightarrow A(n) \equiv 0 \pmod 5 \\ &n \equiv 2,3,5 \pmod 7 &&\Rightarrow A(n) \equiv 0 \pmod 7 \\ &n \equiv 2,3,4,6,9 \pmod{11} &&\Rightarrow A(n) \equiv 0 \pmod{11} \\ &n \equiv \begin{array}{l}4,5,6,7,9,\\ 11,12,15,16,19,21\end{array} \pmod{23} &&\Rightarrow A(n) \equiv 0 \pmod{23}. \end{aligned}$$

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Obviously, these congruences are analogues of those of the j-function. In this note, we give an outline of a proof of the first two relations in (1), that is, we have the following theorem:

Theorem 1.1. *The first two relations in* (1) *are true.*

It should be possible to prove the other three cases. However, for the other cases, some computer calculations that are necessary are not yet completed. We believe that the calculations will be completed in the near future, and the details of the proof will be presented in our forthcoming paper [4].

2. Outline of the proof of Theorem 1.1. For the reader's convenience, we quote from [8] the notion of modular forms and harmonic weak Maass forms for an arbitrary multiplier system (for more information, the reader is referred to [8]). A function $f:\mathbf{H}\to\mathbf{C}$ is called a weakly holomorphic modular form of weight $k/2$ with respect to a congruence subgroup $\Gamma\leq SL_2(\mathbf{Z})$ and a multiplier system ν if the following conditions hold:

(1) f satisfies the modular transformation property:

$$f\left(\frac{a\tau+b}{c\tau+d}\right)=\nu\begin{pmatrix}ab\\cd\end{pmatrix}(c\tau+d)^{\frac{k}{2}}f(\tau).$$

(2) f is holomorphic on $\mathbf{H}$.

(3) f has at most linear exponential growth at the cusps.

We call a function $f:\mathbf{H}\to\mathbf{C}$ a harmonic Maass form if the second condition is replaced by the weaker condition that f is annihilated by the weight $k/2$ hyperbolic Laplacian $\Delta_{\frac{k}{2}}:=-y^2(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})+\frac{ik}{2}(\frac{\partial}{\partial x}+\frac{\partial}{\partial y})$. The η-multiplier $\nu_\eta:SL_2(\mathbf{Z})\to\{z\in\mathbf{C}\mid |z|=1\}$ is defined by

$$\nu_\eta\begin{pmatrix}ab\\cd\end{pmatrix}:=\frac{1}{\sqrt{c\tau+d}}\frac{\eta\left(\frac{a\tau+b}{c\tau+d}\right)}{\eta(\tau)}.$$

The η-multiplier system and the integer powers of them are multiplier systems for all half-integral weights [8].

Let

$$g_{a,b}:=\sum_{\nu\in a+\mathbf{Z}}\nu e^{\pi i\nu^2+2\pi i\nu b},$$

where $a,b\in\mathbf{R}$. We define $\widetilde{\mu}(u;\tau)$ and $\Sigma(z)$ as follows:

$$\widetilde{\mu}(z;\tau):=\mu(z;\tau)+\frac{i}{2}R(0;\tau)$$
$$\widetilde{\Sigma}(z):=8\sum_{z\in\{1/2,\tau/2,(1+\tau)/2\}}\widetilde{\mu}(z;\tau),$$

where $R(0;\tau)$ is defined as

$$R(0;\tau):=\int_{-\bar{\tau}}^{\infty}\frac{g_{\frac{1}{2},\frac{1}{2}}(t)}{\sqrt{-i(t+\tau)}}\,dt.$$

Then it follows from [9] and [8, Lemma 3.3] that the following proposition holds:

Proposition 2.1. *The function $\widetilde{\Sigma}(z)$ is a harmonic weak Maass form of weight $1/2$ with respect to the group $SL_2(\mathbf{Z})$ with multiplier system ν_η^{-3}.*

It follows from [9, Propositions 3.6 & 3.7] that $\mu(1/2;\tau)$, $\mu(\tau/2;\tau)$, and $\mu((1+\tau)/2;\tau)$ are the holomorphic part of $\widetilde{\mu}(1/2;\tau)$, $\widetilde{\mu}(\tau/2;\tau)$, and $\widetilde{\mu}((1+\tau)/2;\tau)$, respectively.

For a harmonic weak Maass form with Fourier expansion

$$f(\tau)=\sum_{n\in\mathbf{Z}}a_n(y)q^{\frac{n}{w}}$$

with $w\in N$ and $r,m\in\mathbf{Z}$, we define the sieve operator by

$$U_{r,m}f(\tau):=\sum_{n\equiv r \pmod m}a_n(y)q^{\frac{n}{w}}.$$

Proposition 2.2 (cf. [8, Proposition 3.8]). *Let f be a harmonic weak Maass form of weight $1/2$ for $\Gamma_0(2)$ with respect to ν_η^{-3}. Suppose that f has a Fourier expansion in terms of $q^{\frac{1}{8}}$. Let $r,m\in\mathbf{N}$. Then $U_{r,m}f$ is a harmonic weak Maass form of weight $1/2$ with respect to the congruence subgroup*

$$\left\{\begin{pmatrix}ab\\cd\end{pmatrix}\in SL_2(\mathbf{Z})\;\middle|\; a,d \text{ coprime to } m,\; a\equiv d \pmod m \text{ and } 2m^2|c\right\}.$$

Also, $\Gamma_1(2m^2)\leq\Gamma\leq\Gamma_0(2m^2)$ and

$$[SL_2(\mathbf{Z}):\Gamma]=\frac{2m^4\varphi(m)}{\varphi(2m^2)}\prod_{p|2m^2}\left(1-\frac{1}{p^2}\right).$$

By Proposition 2.1, the function $\widetilde{\Sigma}(z)$ is a harmonic weak Maass form with respect to the group $\Gamma_0(2)$. Therefore, we can apply Proposition 2.2 for the function $\widetilde{\Sigma}(z)$.

From here, we give the outline of the proof of Theorem 1.1. In particular, we give only the case $n\equiv 1,2 \pmod 3\Rightarrow A(n)\equiv 0 \pmod 3$. The other case can be proved by a similar argument.

Let

$$\Sigma_{7,15,(24)}(\tau):=\sum_{n\equiv 7,15 \pmod{24}}A(n)q^{\frac{n}{8}}.$$

It is sufficient to prove that all the coefficients of $\Sigma_{7,15,(24)}(\tau)$ are multiples of 3. First, it can be proved by a method similar to that of [8, Proposition 3.9] and Proposition 2.2 that $\Sigma_{7,15,(24)}(\tau)$ is a weakly holomorphic modular form of weight 1/2 with respect to ν_η^{-3} for some subgroup Γ of $SL_2(\mathbf{Z})$ that contains $\Gamma_1(1152)$ and has index 9216 in $SL_2(\mathbf{Z})$. Moreover, using [5, Theorem 1.64 & 1.65] and [8, Lemma 3.11], $\eta(24\tau)^{24}\eta(\tau)^{24}$ is a cusp form of weight 24 for $\Gamma_0(1152)$, and $\eta(24\tau)^{24}\eta(\tau)^{24}\Sigma_{7,15,(24)}(\tau)$ is holomorphic at every cusp of $\Gamma_1(1152)$. Then, using Sturm's theorem [7], which states that all the coefficients of a modular form of weight $k/2$ with respect to Γ are divisible by a prime p if and only if the first

$$\left(\frac{k}{24}\right)[SL_2(\mathbf{Z}):\Gamma]$$

coefficients are divisible by p, it is sufficient to check that the first 18816 coefficients are multiples of 3. We have checked numerically that this holds. Since the leading coefficient of $\eta(24\tau)^{24}\eta(\tau)^{24}$ is 1, by induction, all the coefficients of $\Sigma_{7,15,(24)}(\tau)$ are multiples of 3.

Remark 2.1.

(1) For the case "$n \equiv 1,3 \pmod 5 \Rightarrow A(n) \equiv 0 \pmod 5$", we define the function as follows:

$$\Sigma_{7,23,(40)}(\tau) := \sum_{n\equiv 7,23 \ (\mathrm{mod}\ 40)} A(n)q^{\frac{n}{8}}.$$

The function $\eta(40\tau)^{48}\eta(\tau)^{24}\Sigma_{7,23,(40)}(\tau)$ and are holomorphic at every cusp of $\Gamma_1(3200)$. Then it can be proved by a similar argument that checking that the first 140160 coefficients are divisible by 5, is sufficient.

(2) A similar argument can be found in [8]. We note that Proposition 3.12 in [8] has a mistake. The function $\eta(24\tau)^{12}\Delta(\tau)$ should be $\eta(24\tau)^{24}\Delta(\tau)$ because the function $\widetilde{\mathcal{C}}_{7(24)}(\tau)$ appearing in Proposition 3.12 has a pole at the cusp 1/12. Moreover, in [8], Waldherr reported checking that the first 7104 coefficients of $\widetilde{\mathcal{C}}_{7(24)}(\tau)$ are divisible by 3 in the proof of Proposition 3.13. However, what should have been checked are the first 14208 coefficients since $(37/24)\times 9216 = 14208$. By the same reason as above, 260736 appearing in the Proof of Proposition 3.13 should be 521472.

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