Chern classes and generators

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Abstract: We give a simple proof for the fact that certain algebra generators of the mod 2 cohomology of classifying spaces of exceptional Lie groups are given by Chern classes and Stiefel-Whitney classes of certain representations.

Key words: Classifying space; exceptional Lie group; characteristic class.

1. Introduction. One of the standard tools to compute the mod 2 cohomology of the classifying space BG of a connected Lie group G is the Rothenberg-Steenrod spectral sequence converging to the mod 2 cohomology of BG. In the case of $G = E_6$ or E_7, as an algebra over the Steenrod algebra, the E_2-term of the spectral sequence is generated by only two elements, one is of degree 4 and the other is of degree 32 or 64, respectively. To show the collapsing of the spectral sequence, it suffices to show that the algebra generator of degree 32 or 64 survives to the E_∞-term, respectively. It could be done by showing that the algebra generator is represented by a characteristic class of some representation of G. It is also conjectured that the mod 2 cohomology of BE_8 is also generated by two elements as an algebra over the mod 2 Steenrod algebra.

The case $G = E_6$ is first proved by Kono and Mimura in [3], see also [9]. The case $G = E_7$ is proved by Kono, Mimura and Shimada in [4]. The case $G = E_8$ of the following theorem is first proved by Mimura and Nishimoto in [6] with rather complicated calculation. Then, Kono, in [5], gives a simple proof for the following theorem by considering certain finite 2-groups in exceptional Lie groups and computing Stiefel-Whitney classes of their representations.

Theorem 1.1. *For $G = F_4, E_6, E_7, E_8$, there exist representations ρ_4, ρ_6, ρ_7, ρ_8 such that $w_{16}(\rho_4)$, $c_{16}(\rho_6)$, $c_{32}(\rho_7)$ and $w_{128}(\rho_8)$ are indecomposable in $H^*(BG; \mathbf{Z}/2)$.*

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In this paper, we give another simpler proof for this theorem using Chern classes of representations of spinor groups. The method in this paper is used by Schuster, Yagita and the author in [8] and [2], where the degree 4 element in the integral cohomology of the above classifying spaces is studied in conjunction with Chern subrings of classifying spaces.

2. Chern classes. Let us recall complex representation rings of spinor groups. For $n \geq 6$, we consider $m = [\frac{n}{2}]$, so that $n = 2m$ or $2m+1$. Let T^m be a fixed maximal torus of $\mathrm{Spin}(n)$ and denote by $f_m : T^m \to \mathrm{Spin}(n)$ the inclusion. We also denote by $f_1 : T^1 \to \mathrm{Spin}(n)$ the composition of the inclusion of the maximal torus and the inclusion of the first factor T^1 into T^m. The complex representation ring of $\mathrm{Spin}(n)$ is given by

$$R(\mathrm{Spin}(2m)) = \mathbf{Z}[\lambda_1, \dots, \lambda_{m-2}, \Delta^+, \Delta^-],$$

and

$$R(\mathrm{Spin}(2m+1)) = \mathbf{Z}[\lambda_1, \dots, \lambda_{m-1}, \Delta].$$

The image $f_m^*(\lambda_i)$ of the above generator λ_i in

$$R(T^m) = \mathbf{Z}[z_1, z_1^{-1}, \dots, z_m, z_m^{-1}],$$

is the i-th elementary symmetric function of $z_1^2 + z_1^{-2}, \dots, z_m^2 + z_m^{-2}$. The image $f_m^*(\Delta^+)$, $f_m^*(\Delta^-)$, $f_m^*(\Delta)$ in $R(T^m)$ are given by

$$f_m^*(\Delta^+) = \sum_{\varepsilon_1 \cdots \varepsilon_m = +1} z_1^{\varepsilon_1} \dots z_m^{\varepsilon_m},$$

$$f_m^*(\Delta^-) = \sum_{\varepsilon_1 \cdots \varepsilon_m = -1} z_1^{\varepsilon_1} \dots z_m^{\varepsilon_m},$$

and

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$$f_m^*(\Delta) = \sum_{\varepsilon_1 \cdots \varepsilon_m = \pm 1} z_1^{\varepsilon_1} \ldots z_m^{\varepsilon_m}.$$

Since $z_i^2 + z_i^{-2}$ maps to 2, for $1 \le i \le m$, it is clear that

$$f_1^*(\lambda_i) = \alpha_i + \beta_i (z_1^2 + z_1^{-2}),$$

where $\alpha_i = 2^i \binom{m-1}{i}$, $\beta_i = 2^{i-1}\binom{m-1}{i-1}$. It is also clear that

$$\begin{aligned} f_1^*(\Delta^+) &= f_1^*(\Delta^-) \\ &= 2^{m-2}(z_1 + z_1^{-1}), \\ f_1^*(\Delta) &= 2^{m-1}(z_1 + z_1^{-1}). \end{aligned}$$

Therefore, the total Chern classes are

$$\begin{aligned} c(f_1^*(\lambda_i)) &= \{(1+2u)(1-2u)\}^{\beta_i} \\ &= (1-4u^2)^{\beta_i}, \\ c(f_1^*(\Delta^+)) &= c(f_1^*(\Delta^-)) \\ &= \{(1+u)(1-u)\}^{2^{m-2}} \\ &= (1-u^2)^{2^{m-2}}, \\ c(f_1^*(\Delta)) &= \{(1+u)(1-u)\}^{2^{m-1}} \\ &= (1-u^2)^{2^{m-1}} \end{aligned}$$

in $H^*(BT^1; \mathbf{Z}) \cong \mathbf{Z}[u]$, where u is the generator of $H^2(BT^1; \mathbf{Z}) \cong \mathbf{Z}$.

Thus we have the following result on the mod 2 reduction of the total Chern classes:

Proposition 2.1. *The mod 2 reduction of the total Chern classes of $f_1^*(\lambda_i)$, $f_1^*(\Delta^{\pm})$, $f_1^*(\Delta)$ are given by*

$$\begin{aligned} c(f_1^*(\lambda_i)) &= 1, \\ c(f_1^*(\Delta^+)) &= c(f_1^*(\Delta^-)) \\ &= 1 + u^{2^{m-1}}, \\ c(f_1^*(\Delta)) &= 1 + u^{2^m} \end{aligned}$$

in $H^*(BT^1; \mathbf{Z}/2)$. In particular, $c(f_1^*\mu) = 1 + u^k$ where $\deg u^k = 2k = 2\dim \mu$ for $\mu = \Delta^+, \Delta^-, \Delta$.

3. Generators. In this section, let $H^*(X)$ be the mod 2 cohomology $H^*(X; \mathbf{Z}/2)$ of X and $\tilde{H}^*(X)$ the reduced mod 2 cohomology $\tilde{H}^*(X; \mathbf{Z}/2)$. Recall that if ξ is a real representation and if $\xi_{\mathbf{C}}$ is its complexification, then, the mod 2 Chern class $c_i(\xi_{\mathbf{C}})$ is the square of the Stiefel-Whitney class $w_i(\xi)$, that is, $c_i(\xi_{\mathbf{C}}) = w_i(\xi)^2$.

We recall Quillen's computation of the mod 2 cohomology of $B\mathrm{Spin}(n)$ in [7]. As in the previous section, let $m = [\frac{n}{2}]$. Depending on the type of the representation of the spinor group $\mathrm{Spin}(n)$, we define h to be $m-1$ or m in Table I.

Table I.

n	m	type	h	h
$8k$	$4k$	$\mathbf{R}$	$4k-1$	$m-1$
$8k+1$	$4k$	$\mathbf{R}$	$4k-1$	$m-1$
$8k+2$	$4k+1$	$\mathbf{C}$	$4k+1$	m
$8k+3$	$4k+1$	$\mathbf{H}$	$4k+1$	m
$8k+4$	$4k+2$	$\mathbf{H}$	$4k+2$	m
$8k+5$	$4k+2$	$\mathbf{H}$	$4k+2$	m
$8k+6$	$4k+3$	$\mathbf{C}$	$4k+3$	m
$8k+7$	$4k+3$	$\mathbf{R}$	$4k+2$	$m-1$

Let $p : \mathrm{Spin}(n) \to SO(n)$ be the projection. Then, the mod 2 cohomology of $B\mathrm{Spin}(n)$ is given by

$$H^*(B\mathrm{Spin}(n)) = H^*(BSO(n))/J \otimes \mathbf{Z}/2[z],$$

where J is the ideal generated by $w_2, \mathrm{Sq}^1 w_2, \ldots, \mathrm{Sq}^{2^{h-2}} \cdots \mathrm{Sq}^1 w_2$ and z is an element of degree 2^h. Let $f_1 : T^1 \to \mathrm{Spin}(n)$ be the inclusion of the first factor of the maximal torus. Since T^1 is a closed subgroup of $\mathrm{Spin}(n)$, the induced homomorphism $Bf_1^* : H^*(B\mathrm{Spin}(n)) \to H^*(BT^1)$ is integral. On the other hand, since $c(f_1^*(\lambda_1)) = 1$, the induced homomorphism $(Bp \circ Bf_1)^* : \tilde{H}^*(BSO(n)) \to \tilde{H}^*(BT^1)$ is zero. Hence, $Bf_1^*(z)$ must be non-zero, that is, $Bf_1^*(z) = u^{2^{h-1}}$ and it generates the image of Bf_1^*.

Thus, we have the following proposition:

Proposition 3.1. *The image of Bf_1^* is a polynomial ring generated by $u^{2^{h-1}}$.*

Now, we consider the following representations for exceptional Lie groups F_4, E_6, E_7, E_8:

$$\begin{array}{lllllll} \mathrm{Spin}(9) & \xrightarrow{g_4} & F_4 & \xrightarrow{\rho_4} & SO(26) & \longrightarrow & SU(26), \\ \mathrm{Spin}(10) & \xrightarrow{g_6} & E_6 & \xrightarrow{\rho_6} & SU(27), & & \\ \mathrm{Spin}(12) & \xrightarrow{g_7} & E_7 & \xrightarrow{\rho_7} & Sp(28) & \longrightarrow & SU(56), \\ \mathrm{Spin}(16) & \xrightarrow{g_8} & E_8 & \xrightarrow{\rho_8} & SO(248) & \longrightarrow & SU(248) \end{array}$$

such that

$$\begin{aligned} g_4^*(\rho_4) &= 1 + \lambda_1 + \Delta, \\ g_6^*(\rho_6) &= 1 + \lambda_1 + \Delta^+, \\ g_7^*(\rho_7) &= 2\lambda_1 + \Delta^-, \\ g_8^*(\rho_8) &= 8 + \lambda_2 + \Delta^+. \end{aligned}$$

The existence of such representations (and their construction) is proved in Adams' book [1]. These representations are tightly connected with the construction of exceptional Lie groups.

Table II.

G	Spin(n)	m	rep.	dim of rep.	$\deg z = 2^h$
F_4	Spin(9)	4	Δ	16	16
E_6	Spin(10)	5	Δ^+	16	32
E_7	Spin(12)	6	Δ^-	32	64
E_8	Spin(16)	8	Δ^+	128	128

With the following Table II, we summarize the information we need in the proof of Theorem 1.1.

Proof of Theorem 1.1. For $G = E_6$, E_7, by Proposition 2.1, we have

$$c_{16}(f_1^* g_6^* \rho_6) = u^{16}, \quad c_{32}(f_1^* g_7^* \rho_7) = u^{32}.$$

By Proposition 3.1, these elements are indecomposable in $\mathrm{Im}\, Bf_1^*$. So, $c_{16}(\rho_6)$, $c_{32}(\rho_7)$ are indecomposable in $H^*(BE_6)$, $H^*(BE_7)$.

For $G = F_4, E_8$, let $\rho_{4,\mathbf{C}}$, $\rho_{8,\mathbf{C}}$ be complexification of ρ_4, ρ_8, respectively. Then, by Proposition 2.1, we have

$$c_{16}(f_1^* g_4^* \rho_{4,\mathbf{C}}) = u^{16}, \quad c_{128}(f_1^* g_8^* \rho_{8,\mathbf{C}}) = u^{128}.$$

These elements are decomposable in $\mathrm{Im}\, Bf_1^*$. However, since $\rho_{4,\mathbf{C}}$, $\rho_{8,\mathbf{C}}$ are complexification of ρ_4, ρ_8, we have $w_{16}(f_1^* g_4^* \rho_4) = u^8$, $w_{128}(f_1^* g_8^* \rho_8) = u^{64}$. By Proposition 3.1, these elements are indecomposable in $\mathrm{Im}\, Bf_1^*$. Hence, $w_{16}(\rho_4)$, $w_{128}(\rho_8)$ are indecomposable in $H^*(BF_4)$, $H^*(BE_8)$. □

Remark 3.1. If it is shown that, using the Rothenberg-Steenrod spectral sequence, the generators of the mod 2 cohomology of BG is generated by two elements and that one of these two generators is the degree 4 element, say y_4, then since $f_1^*(g_i^*(y_4)) = 0$, we have that $\mathrm{Im}\, f_1^* \circ g_i^*$ is generated by $f_1^*(g_4^*(w_{16}(\rho_4)))$, $f_1^*(g_6^*(c_{16}(\rho_6)))$, $f_1^*(g_7^*(c_{32}(\rho_7)))$, $f_1^*(g_8^*(w_{128}(\rho_8)))$, respectively. Thus, we do not need to refer the reader to Quillen's computation of the mod 2 cohomology of $B\mathrm{Spin}(n)$ to complete the proof of Theorem 1.1. This is the case for $G = F_4, E_6, E_7$ and, if, for some r, E_r-term of the spectral sequence converging to the mod 2 cohomology of BE_8 is also generated by two elements as an algebra over the Steenrod algebra, this argument is also applicable to the case $G = E_8$.

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