Defect zero characters and relative defect zero characters

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Abstract: For a normal subgroup K of a finite group G and a G-invariant irreducible character ξ of K we show under a certain condition there is a bijection between the set of relative defect zero irreducible characters of G lying over ξ and the set of defect zero irreducible characters of G/K.

Key words: Defect zero character; relative defect zero character; blocks with central defect groups.

1. Introduction. Let G be a finite group and p a prime. Let $(\mathcal{K}, R, k)$ be a p-modular system ([NT, p.230]). We assume $\mathcal{K}$ contains a primitive $|G|^2$-th root of unity. After [Is, p.186] we say (G, K, ξ) a character triple, if K is a normal subgroup of G and ξ is a G-invariant irreducible character of K. Let (G, K, ξ) be a character triple. As in [Na], let $\mathrm{dz}(G/K)$ be the set of irreducible characters of G/K of p-defect 0 and let $\mathrm{rdz}(G|\xi)$ be the set of irreducible characters χ of G lying over ξ such that $(\chi(1)/\xi(1))_p = |G/K|_p$.

Let $\mathcal{K}_0$ be the algebraic closure of the prime field $\mathbf{Q}$ in $\mathcal{K}$. As in [NT, p.230], we regard $\mathcal{K}_0$ as a subfield of the field of complex numbers. We introduce the following

Definition. Let (G, K, ξ) be a character triple. A $\mathcal{K}_0$-valued class function $\tilde{\xi}$ on G is said to be a p-quasi extension of ξ to G if $\tilde{\xi}_L$ is an extension (as a character) of ξ for any subgroup L of G such that $L \geq K$ and that L/K is a p'-group.

For the character triple (G, K, ξ), a cohomology class of G/K (an element of $H^2(G/K, \overline{\mathcal{K}}^{\times})$, where $\overline{\mathcal{K}}$ is the algebraic closure of $\mathcal{K}$) associated to ξ is defined by [Is, Theorem 11.7], which we denote by $\omega_{G/K}(\xi)$. The purpose of this note is to prove the following

Theorem. *Let (G, K, ξ) be a character triple. Then it holds the following.*

(1) ξ has a p-quasi extension to G if and only if $\omega_{G/K}(\xi)$ has p-power order.

(2) Assume that one of the conditions in (1) holds. Then for any p-quasi extension $\tilde{\xi}$ of ξ to G, the map sending θ to $\tilde{\xi}\theta$ is a bijection from $\mathrm{dz}(G/K)$ onto $\mathrm{rdz}(G|\xi)$.

(3) Such a map in (2) is determined uniquely by a linear character of G/K.

2. Proof of Theorem. Let ν be as in [NT, p.230].

Proposition 1. *Let (G, K, ξ) be a character triple. If $\tilde{\xi}$ is a p-quasi extension of ξ to G, then the map sending θ to $\tilde{\xi}\theta$ is a bijection of $\mathrm{dz}(G/K)$ onto $\mathrm{rdz}(G|\xi)$. In particular, $|\mathrm{dz}(G/K)| = |\mathrm{rdz}(G|\xi)|$.*

Proof. We first show that $\tilde{\xi}\theta$ is a generalized character by using Brauer's theorem ([Fe, Theorem IV 1.1], [NT, Theorem 3.4.2]). Let E be an elementary subgroup of G. It suffices to show $(\tilde{\xi}\theta)_{EK}$ is a generalized character. Let η be an irreducible character of EK. Since EK/K is nilpotent there exist a subgroup M with $EK \geq M \geq K$ and a character ϕ of M such that ϕ_K is irreducible and that $\phi^{EK} = \eta$ by [Is, Theorem 6.22]. Put $\overline{G} = G/K$ and use the bar convention. Put $L/K = \mathrm{O}^p(M/K)$. We have

$$\begin{aligned}(\tilde{\xi}\theta, \eta)_{EK} &= (\tilde{\xi}\theta, \phi)_M \\ &= \frac{1}{|M|}\sum_{x\in L}\tilde{\xi}(x)\theta(x)\overline{\phi(x)} \\ &= \frac{1}{|M|}\sum_{\overline{x}\in\overline{L}}\theta(\overline{x})\sum_{y\in xK}\tilde{\xi}_L(y)\overline{\phi_L(y)}.\end{aligned}$$

The inner sum equals 0 if $\phi_K \neq \xi$ by [Is, Lemma 8.14]. So we may assume $\phi_K = \xi$. Then both ϕ_L and $\tilde{\xi}_L$ are extensions of ξ to L. Hence there is a linear character ψ of L/K such that $\phi_L \otimes \psi = \tilde{\xi}_L$. Thus the above sum equals by [Is, Lemma 8.14]

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$$\frac{1}{|M|}\sum_{\overline{x}\in\overline{L}}\theta(\overline{x})\sum_{y\in xK}|\phi_L(y)|^2\psi(y) = \frac{1}{|\overline{M}|}\sum_{\overline{x}\in\overline{L}}\theta(\overline{x})\psi(\overline{x}) = \frac{|\overline{L}|}{|\overline{M}|}(\theta,\psi)_{\overline{L}} = \frac{n}{|\overline{M}|_p},$$

for some integer n. On the other hand, let $\overline{Q}$ be the Sylow p-subgroup of $\overline{M}$. Then, since θ has p-defect 0, we have $\nu(\theta(\overline{x})) \geq \nu(|C_{\overline{G}}(\overline{x})|) \geq \nu(|\overline{Q}|)$ for $\overline{x}\in\overline{L}$ by [NT, Exercise 6.26, p.245]. Hence

$$\frac{1}{|\overline{M}|}\sum_{\overline{x}\in\overline{L}}\theta(\overline{x})\psi(\overline{x}) = \frac{1}{|\overline{L}|}\sum_{\overline{x}\in\overline{L}}\frac{\theta(\overline{x})}{|\overline{Q}|}\psi(\overline{x})$$

is a local integer. Thus $(\tilde{\xi}\theta,\eta)_{EK}$ is an integer, as required.

Next we want to show $(\tilde{\xi}\theta,\tilde{\xi}\theta')_G = \delta_{\theta\theta'}$ (Kronecker delta) for $\theta,\theta'\in \mathrm{dz}(G/K)$. Let $\overline{G}_{p'}$ be the set of p'-elements of $\overline{G}$. We have, by [Is, Lemma 8.14],

$$\begin{aligned}(\tilde{\xi}\theta,\tilde{\xi}\theta')_G &= \frac{1}{|G|}\sum_{y\in G}|\tilde{\xi}(y)|^2\theta(y)\overline{\theta'(y)}\\ &= \frac{1}{|G|}\sum_{\overline{x}\in\overline{G}_{p'}}\sum_{y\in xK}|\tilde{\xi}_{\langle x,K\rangle}(y)|^2\theta(\overline{x})\overline{\theta'(\overline{x})}\\ &= \frac{1}{|\overline{G}|}\sum_{\overline{x}\in\overline{G}_{p'}}\theta(\overline{x})\overline{\theta'(\overline{x})}\\ &= \delta_{\theta\theta'}.\end{aligned}$$

Since $(\tilde{\xi}\theta)(1) > 0$, $\tilde{\xi}\theta$ is an irreducible character. Clearly $\tilde{\xi}\theta\in \mathrm{rdz}(G|\xi)$. Thus the map sending $\theta\in \mathrm{dz}(\overline{G})$ to $\tilde{\xi}\theta\in\mathrm{rdz}(G|\xi)$ is a well-defined injection. We will prove below that $|\mathrm{rdz}(G|\xi)| = |\mathrm{dz}(G/K)|$. Then the map is a bijection. The proof is complete. □

Example. Let (G,K,ξ) be a character triple such that K is a p-group. For any subgroup L of G such that $L\geq K$ and that L/K is a p'-group, there is a canonical extension $\hat{\xi}(L)$ of ξ to L by [Is, Corollary 8.16]. Namely, $\hat{\xi}(L)$ is a unique extension of ξ to L such that $\det(\hat{\xi}(L))$ has p-power order. Define $\tilde{\xi}$ by $\tilde{\xi}(x) = \hat{\xi}(\langle x,K\rangle)(x)$ if xK is a p'-element of G/K, $\tilde{\xi}(x) = 0$ otherwise. Then for any L as above $\tilde{\xi}_L = \hat{\xi}(L)$ by uniqueness of canonical extension. Thus $\tilde{\xi}$ is a p-quasi extension of ξ to G. Hence Proposition 1 gives (most of) Theorem 2.1 of [Na].

Proposition 2. *Let (G,K,ξ) be a character triple. The following are equivalent.*

(i) *ξ has a p-quasi extension to G.*

(ii) *ξ is extendible to any subgroup L of G such that $L\geq K$ and that L/K is a p'-group.*

(iii) *The cohomology class $\omega_{G/K}(\xi)$ has p-power order.*

Proof. (i)$\Longrightarrow$(ii): Trivial by definition.

(ii)$\Longleftrightarrow$(iii): By cohomology theory, cf. [NT, Problem 10, p.164].

(iii)$\Longrightarrow$(i): Let p^n be the order of $\omega_{G/K}(\xi)$. There is a central extension of G

$$1\longrightarrow Z\longrightarrow \hat{G}\xrightarrow{\ f\ } G\longrightarrow 1$$

with the following properties: for some $K_1\lhd\hat{G}$, $f^{-1}(K) = K_1\times Z$, ξ extends to a character $\hat{\xi}$ of an irreducible $\overline{\mathcal{K}}\hat{G}$-module (we identify K_1 with K via f), and Z is a cyclic group of order p^n.

Since p^n divides $|G/K|$ and $\mathcal{K}$ contains a primitive $|G|^2$-th root of unity, $\mathcal{K}$ contains a primitive $|\hat{G}|$-th root of unity. Hence $\hat{\xi}$ is a character of irreducible $\mathcal{K}\hat{G}$-module. Let λ be an irreducible constituent of $\hat{\xi}_Z$. Define a linear character λ^* of $K\times Z$ by $\lambda^* = 1_K\times\lambda$. Define a function $\tilde{\xi}$ on G by:

$$\begin{aligned}\tilde{\xi}(x) &= \hat{\xi}(\hat{x})\lambda^*(\hat{x}_p)^{-1} \quad \text{if } x_p\in K\\ &= 0 \quad \text{if } x_p\notin K\end{aligned}$$

where $x\in G$, x_p is the p-part of x and $\hat{x}$ is an element of $\hat{G}$ such that $f(\hat{x}) = x$. If $x_p\in K$, then $\hat{x}_p\in K\times Z$. Thus the definition makes sense. We show that $\tilde{\xi}$ is well-defined. It suffices to consider the case where $x_p\in K$. Let $\hat{x}' = \hat{x}z$ for $z\in Z$. Then $\hat{\xi}(\hat{x}')\lambda^*(\hat{x}'_p)^{-1} = \hat{\xi}(\hat{x}z)\lambda^*(\hat{x}_pz)^{-1} = \hat{\xi}(\hat{x})\lambda(z)\lambda^*(\hat{x}_p)^{-1}\lambda^*(z)^{-1} = \hat{\xi}(\hat{x})\lambda^*(\hat{x}_p)^{-1}$, as required. We show that $\tilde{\xi}$ is a p-quasi extension of ξ to G. It is easy to see that $\tilde{\xi}$ is a $\mathcal{K}_0$-valued class function on G. Let L be any subgroup of G such that $L\geq K$ and that L/K is a p'-group. By cohomology theory there is an extension ξ^* of ξ to L. We show there is a linear character ψ of L/K such that $\xi^*\otimes\psi = \tilde{\xi}_L$. Put $\hat{L} = f^{-1}(L)$. Then $\mathrm{Inf}_{L\to\hat{L}}\xi^* = \hat{\xi}_{\hat{L}}\otimes\mu$ for a linear character μ of $\hat{L}/K$, where $\mathrm{Inf}_{L\to\hat{L}}\xi^*$ is the inflation of ξ^* to $\hat{L}$. Then $\lambda\mu_Z = 1_Z$. Define a function $\hat{\psi}$ on $\hat{L}$ by $\hat{\psi}(\hat{x}) = (\mu(\hat{x})\lambda^*(\hat{x}_p))^{-1}$ for $\hat{x}\in\hat{L}$. Then $\hat{\psi}$ is a linear character of $\hat{L}$. Indeed, let $\hat{x},\hat{y}\in\hat{L}$. $\hat{L}/K$ has a central Sylow p-subgroup KZ/K, so that $\hat{x}_p\hat{y}_p\equiv(\hat{x}\hat{y})_p \bmod K$. Further λ^* is trivial on K. Hence $\lambda^*(\hat{x}_p)\lambda^*(\hat{y}_p) = \lambda^*((\hat{x}\hat{y})_p)$. Therefore $\hat{\psi}(\hat{x})\hat{\psi}(\hat{y}) = \hat{\psi}(\hat{x}\hat{y})$, as required. It is easy to see $\hat{\psi}$ is trivial on KZ. Hence $\hat{\psi}$ is regarded as a linear character ψ of $L/K\simeq\hat{L}/KZ$. Then for $x\in L$, we have $(\xi^*\otimes\psi)(x) = \tilde{\xi}(x)$. Thus (i) follows.

It remains to prove that if ξ has a p-quasi extension to G, then $|\mathrm{dz}(G/K)| = |\mathrm{rdz}(G|\xi)|$. To prove this we use the p-quasi extension $\tilde{\xi}$ constructed above. In the proof of Proposition 1 we have already proved the map sending θ to $\tilde{\xi}\theta$ is an injection from $\mathrm{dz}(G/K)$ to $\mathrm{rdz}(G|\xi)$. Therefore, to prove $|\mathrm{dz}(G/K)| = |\mathrm{rdz}(G|\xi)|$, it suffices to prove this map is a surjection. Let $\chi \in \mathrm{rdz}(G|\xi)$ and put $\tilde{G} = \hat{G}/K$ and $\tilde{Z} = ZK/K$. Then $\mathrm{Inf}_{G\to\hat{G}}\chi = \hat{\xi} \otimes \tilde{\chi}$ for some irreducible character $\tilde{\chi}$ of $\tilde{G}$. Then $\nu(\tilde{\chi}(1)) = \nu(|G/K|) = \nu(|\tilde{G}/\tilde{Z}|)$. Let $\tilde{B}$ be the p-block of $\tilde{G}$ containing $\tilde{\chi}$. Then $\tilde{Z}$ is a defect group of $\tilde{B}$ by [La] (see also [Mu]). Via the natural isomorphism $Z \simeq \tilde{Z}$, λ may be regarded as a linear character of $\tilde{Z}$. Since λ^{-1} is an irreducible constituent of $\tilde{\chi}_{\tilde{Z}}$, we obtain the value of $\tilde{\chi}$:

$$\begin{aligned}\tilde{\chi}(\tilde{x}) &= \lambda^{-1}(\tilde{x}_p)\tilde{\theta}(\tilde{x}) \quad \text{if } \tilde{x}_p \in \tilde{Z},\\ &= 0 \quad \text{if } \tilde{x}_p \notin \tilde{Z}\end{aligned}$$

where $\tilde{\theta}$ is the canonical character of $\tilde{B}$ by [NT, Theorem 5.8.14]. Since $\tilde{\theta}$ is an irreducible character of $\tilde{G}/\tilde{Z}$ of p-defect 0 and $\tilde{G}/\tilde{Z} \simeq G/K$, $\tilde{\theta}$ may be regarded as an irreducible character θ of G/K of p-defect 0. Then $\theta(x) = \tilde{\theta}(\tilde{x})$ for all $x \in G$, where $f(\hat{x}) = x$, $\hat{x} \in \hat{G}$ and $\tilde{x} = \hat{x}K \in \tilde{G}$. Then $\chi(x) = (\mathrm{Inf}_{G\to\hat{G}}\chi)(\hat{x}) = \hat{\xi}(\hat{x})\tilde{\chi}(\tilde{x})$. Further $\tilde{x}_p \in \tilde{Z}$ iff $\hat{x}_p \in K \times Z$ iff $x_p \in K$. Hence if $x_p \notin K$, then $\chi(x) = 0 = (\tilde{\xi}\theta)(x)$. If $x_p \in K$, then $\chi(x) = \hat{\xi}(\hat{x})\lambda^{-1}(\tilde{x}_p)\tilde{\theta}(\tilde{x}) = \hat{\xi}(\hat{x})\lambda^{-1}(\tilde{x}_p)\theta(x) = (\tilde{\xi}\theta)(x)$. Thus $\chi = \tilde{\xi}\theta$. The proof is complete. □

We say a p-quasi extension $\tilde{\xi}$ normalized if $\tilde{\xi}(x) = 0$ for all $x \in G$ such that xK is not a p'-element of G/K.

Put

$$\begin{aligned}\tilde{\xi}_n(x) &= \tilde{\xi}(x) \quad \text{if } xK \text{ is a } p'\text{-element of } G/K,\\ &= 0 \quad \text{otherwise.}\end{aligned}$$

Then $\tilde{\xi}_n$ is a normalized p-quasi extension of ξ. Since $\tilde{\xi}\theta = \tilde{\xi}_n\theta$ for any $\theta \in \mathrm{dz}(G/K)$, when we consider the map in Theorem, it suffices to consider normalized p-quasi extensions.

Proposition 3. *Let $\tilde{\xi}$ and $\tilde{\xi}'$ be two normalized p-quasi extensions of ξ to G. Then there is a linear character η of G/K such that $\tilde{\xi}' = \tilde{\xi}\eta$.*

Proof. For any p'-subgroup $\overline{L} = L/K$ of $\overline{G} = G/K$, there is a unique linear character $\lambda(\overline{L})$ of $\overline{L}$ such that $\tilde{\xi}'_L = \tilde{\xi}_L \otimes \lambda(\overline{L})$. For any p'-element $\overline{x}$ of $\overline{G}$, define $\mu(\overline{x}) = \lambda(\overline{\langle x, K\rangle})(x)$. Then if $\overline{L}$ is a p'-subgroup and $\overline{x} \in \overline{L}$, then $\lambda(\overline{L})(\overline{x}) = \mu(\overline{x})$. μ is a class function of $\overline{G}$ defined on $\overline{G}_{p'}$. Indeed, for $y \in \langle x, K\rangle := L$ and $g \in G$, we have $\tilde{\xi}'(y) = \tilde{\xi}(y)\lambda(\overline{L})(y)$ and $\tilde{\xi}'^g(y^g) = \tilde{\xi}^g(y^g)\lambda(\overline{L})^g(y^g)$. Since $\tilde{\xi}'$ and $\tilde{\xi}$ are class functions, we obtain $\tilde{\xi}'^g(y^g) = \tilde{\xi}'(y^g)$ and $\tilde{\xi}^g(y^g) = \tilde{\xi}(y^g)$. Thus $\tilde{\xi}'_{L^g} = \tilde{\xi}_{L^g} \otimes \lambda(\overline{L})^g$. Further, $\tilde{\xi}'_{L^g} = \tilde{\xi}_{L^g} \otimes \lambda(\overline{L^g})$. Hence $\lambda(\overline{L^g}) = \lambda(\overline{L})^g$ by uniqueness. Therefore $\mu(\overline{x^g}) = \lambda(\overline{L^g})(x^g) = \lambda(\overline{L})^g(x^g) = \lambda(\overline{L})(x) = \mu(\overline{x})$.

Put $H = \overline{G}$. Define $\eta(h) = \mu(h_{p'})$ for $h \in H$. Then η is a $\mathcal{K}_0$-valued class function on H and $(\eta, \eta)_H = 1$. Let $E = E_p \times E_{p'}$ be an elementary subgroup of H, where E_p and $E_{p'}$ are respectively the Sylow p-subgroup and the p-complement of E. Let α be a linear character of E. Then $(\eta, \alpha)_E = (1_{E_p}, \alpha)_{E_p}(\lambda(E_{p'}), \alpha)_{E_{p'}}$ is an integer. Then η is a linear character of H by Brauer's theorem [NT, Theorem 3.4.2]. We have $\tilde{\xi}'(x) = (\tilde{\xi}\eta)(x)$ if $\overline{x}$ is a p'-element. Since $\tilde{\xi}'$ and $\tilde{\xi}$ are normalized the result follows. □

Proof of Theorem. The first and second assertions of Theorem follow from Propositions 1 and 2. For the last assertion, as remarked above it suffices to consider normalized one. So Proposition 3 yields the result. The proof is complete. □

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