Solvability of primitive equations for the ocean with vertical mixing

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Abstract: Small time existence of a strong solution to the free surface problem of primitive equations for the ocean with the variable turbulent viscosity terms is shown in this paper. The turbulent viscosity coefficients, which include the Richardson number depending on the unknown variables, are formulated explicitly. We consider the problem in the 3-dimensional stripe-like region, and construct the strong local-in-time solution in Sobolev-Slobodetskiĭ spaces. The details of the proofs will be provided in another full paper.

Key words: Primitive equations; Sobolev-Slobodetskiĭ space; strong solution.

1. Introduction. In the present paper, we investigate a free surface problem of primitive equations taking the vertical mixing into account, and show the unique existence of a strong local-in-time solution, which is a new result developed from our earlier result [3]. We adopt practical boundary conditions subject to Bryan [1], Cox [2] and Killworth [4]. Another feature of the present work is taking the parametrization of the vertical mixing into account. All of the existing results in mathematics ([5], for instance) regarded the turbulent viscosity coefficients to be positive constants, while in the parametrization of oceanography, they often depend on the horizontal velocity and the temperature. This leads to the difficulty in the estimate of the principal terms. Furthermore, for the equation of state, we adopt a general form $\varrho = \varrho(p, \theta, S)$ here.

2. Formulation of the problem. Our problem is formulated in the 3-dimensional strip-like region. By $x = (x_1, x_2, x_3)$, we denote an orthogonal Cartesian coordinate system with x_3 being the vertical direction. Let the unknown free surface and the known bottom of the ocean be represented by $x_3 = F(x', t)$ and $x_3 = b(x')$ $(x' = (x_1, x_2))$, respectively. The initial value $F_0(x')$ of $F(x', t)$ is assumed to satisfy $F_0(x') - b(x') > c_0$ with a positive constant c_0 for any $x' \in \mathbf{R}^2$. Then the domain $\Omega(t)$ of the ocean at time t is represented as $\{(x', x_3) | x' \in \mathbf{R}^2, b(x') < x_3 < F(x', t)\}$. Making use of the Boussinesq approximation, the equations that we consider in the present paper are as follows:

$$
(2.1)\quad \begin{cases} \dfrac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} - \left[\mu_1 \Delta \mathbf{v} + \mu_2 \dfrac{\partial^2 \mathbf{v}}{\partial x_3^2}\right] + f\mathbf{A}\mathbf{v} = -\dfrac{1}{\varrho_0}\nabla p, \\ \dfrac{\partial p}{\partial x_3} = -\varrho g, \quad \nabla \cdot \mathbf{v} + \dfrac{\partial w}{\partial x_3} = 0, \\ \dfrac{\mathrm{D}\theta}{\mathrm{D}t} - \left[\mu_3 \Delta \theta + \mu_4 \dfrac{\partial^2 \theta}{\partial x_3^2}\right] = 0, \\ \dfrac{\mathrm{D}S}{\mathrm{D}t} - \left[\mu_5 \Delta S + \mu_6 \dfrac{\partial^2 S}{\partial x_3^2}\right] = 0, \\ \varrho = \varrho(p, \theta, S) \quad x \in \Omega(t),\ t > 0. \end{cases}
$$

Here, $f\mathbf{A}\mathbf{v}$ is a Coriolis force with $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and the Coriolis parameter f is a positive constant due to the f-approximation; ∇ and Δ are 2-dimensional gradient and Laplacian, respectively. The horizontal and vertical components of the velocity are represented by $\mathbf{v}$ w, respectively; p, the pressure; ϱ, the density; ϱ_0, a positive constant; g, the gravity force (a positive constant); θ, the temperature; S, the salinity; (μ_1, μ_2), coefficients of the turbulent viscosity; (μ_3, μ_4) and (μ_5, μ_6), scaled sums of turbulent and molecular diffusivities, respectively, μ_i $(i = 1, 3, 5)$ are positive constants. In addition, we assume ([1], [4], [7])

$$
(2.2)\qquad \frac{\mathrm{D}}{\mathrm{D}t}(x_3 - F(x', t)) = 0,
$$

where $\dfrac{\mathrm{D}}{\mathrm{D}t} \equiv \dfrac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + w \dfrac{\partial}{\partial x_3}$ is an operator known as the material derivative. Following [6] and [7],

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$$\mu_i = \mu_i\left(\frac{\partial \mathbf{v}}{\partial x_3}, \varrho, \frac{\partial \varrho}{\partial x_3}\right) = \mu_{ia}\left(1+\alpha_i \mathcal{R}\left(\frac{\partial \mathbf{v}}{\partial x_3}, \varrho, \frac{\partial \varrho}{\partial x_3}\right)\right)^{-\eta_i} + \mu_{ib},$$

$$\mathcal{R} = \mathcal{R}\left(\frac{\partial \mathbf{v}}{\partial x_3}, \varrho, \frac{\partial \varrho}{\partial x_3}\right) = g\varrho^{-1}\frac{\partial \varrho}{\partial x_3}\left|\frac{\partial \mathbf{v}}{\partial x_3}\right|^{-2},$$

where $\eta_i = 2$ for $i = 2$, and $\eta_i = 1$ for $i = 4, 6$, μ_{ia}, μ_{ib} $(i = 2, 4, 6)$ are positive constants, and $\mathcal{R}$ is the Richardson number. Conditions on the free surface $\Gamma(t) = \{x \in \mathbf{R}^3 | x_3 = d(x', t)\}$ and $\Gamma_b = \{x', b(x') | x' \in \mathbf{R}^2\}$ are:

$$(2.3)\quad \begin{cases} \mu_2 \dfrac{\partial \mathbf{v}}{\partial \mathbf{n}_F} = \tau_1, \ \mu_4 \dfrac{\partial \theta}{\partial \mathbf{n}_F} = \tau_2, \\ \mu_6 \dfrac{\partial S}{\partial \mathbf{n}_F} = g_1 S, \ p = p_0 \quad x \in \Gamma(t), \ t > 0. \end{cases}$$

$$(2.4)\quad \begin{cases} \mu_2 \dfrac{\partial \mathbf{v}}{\partial \mathbf{n}_b} = \mu_4 \dfrac{\partial \theta}{\partial \mathbf{n}_b} = \mu_6 \dfrac{\partial S}{\partial \mathbf{n}_b} = 0, \\ \mathbf{v}\cdot\nabla b + w = 0, \quad x \in \Gamma_b, t > 0. \end{cases}$$

Here $\mathbf{n}_F = (n_1, n_2, n_3)^{\mathrm{T}}$ is the unit normal vector to $\Gamma(t)$ at time t pointing to the ocean region, $\mathbf{n}'_F = (n_1, n_2)^{\mathrm{T}}$; τ_1, the wind shear due to the movement of the atmosphere over the ocean surface; τ_2, the heat flux on the ocean surface; $p_0(x,t)$, the atmospheric pressure at the ocean surface; $g_1(x,t)$, a given function representing the difference of the precipitation and evaporation rate; and $\mathbf{n}_b$, the unit outer normal to Γ_b. Initial conditions on $\Omega \equiv \Omega(0)$ and $\mathbf{R}^2$ are

$$(2.5)\quad \begin{cases} (\mathbf{v}, \theta, S)|_{t=0} = (\mathbf{v}_0, \theta_0, S_0)(x), \quad x \in \Omega, \\ F(x', 0) = F_0(x'), \quad x' \in \mathbf{R}^2, \end{cases}$$

where $\mathbf{v}_0 = (v_{01}, v_{02})^{\mathrm{T}}$. We apply the transform $\Phi_F : (x,t) \longmapsto (y, t^*)$:

$$(2.6)\quad \begin{cases} y' = x', \quad t^* = t, \\ y_3 = (b(x') - F_0(x'))\dfrac{x_3 - F(x',t)}{b(x') - F(x',t)} \\ \quad +F_0(x'). \end{cases}$$

Hereafter we use notations $\widetilde{\Omega}_T \equiv \widetilde{\Omega} \times (0,T)$, $\widetilde{\Gamma}_{bT} \equiv \widetilde{\Gamma}_b \times (0,T)$, $\widetilde{\Gamma}_T \equiv \widetilde{\Gamma} \times (0,T)$, respectively, where

$$\widetilde{\Omega} = \{(y', y_3) | y' \in \mathbf{R}^2, \ b(y') < y_3 < F_0(y')\},$$
$$\widetilde{\Gamma}_b = \{(y', y_3) | y' \in \mathbf{R}^2, \ y_3 = b(y')\},$$
$$\widetilde{\Gamma} = \{(y', y_3) | y' \in \mathbf{R}^2, \ y_3 = F_0(y')\}.$$

In the following, we denote the inverse of the transposed matrix of the Jacobian matrix of Φ_F by

$$(J[(x/y)]^{\mathrm{T}})^{-1} = (a^{ij}) = (a^{ij}(F)) \quad (i, j = 1, 2, 3),$$

and use the notations $\tilde{f}^{(F)}(y,t) \equiv f(\Phi_F^{-1}(y,t))$,

$$\left(\nabla_F, \ \nabla_{F,3}\right)^{\mathrm{T}} = (J[(x/y)]^{\mathrm{T}})^{-1}\left(\nabla_{y'}, \ \frac{\partial}{\partial y_3}\right)^{\mathrm{T}},$$

$$A_1(y,t) \equiv \frac{\partial y}{\partial t} = \frac{y_3 - b(y')}{b(y') - F(y',t)}\frac{\partial F}{\partial t}(y', t),$$

where $\nabla_{y'}$ is the derivative with respect to y'. For a function defined in the whole space $\mathbf{R}^3$, we use the same notation to the one restricted on $\Omega(t)$ at each t and then transfomed into the new coordinate system.

3. Function spaces. Let G be a domain in $\mathbf{R}^n$ $(n = 2, 3)$. By $W_2^l(G)$ we mean a space of functions $u(x)$, $x \in G$, equipped with the norm $\|u\|^2_{W_2^l(G)} = \sum_{|\alpha|<l} \|D^\alpha u\|^2_{L_2(G)} + \|u\|^2_{\dot{W}_2^l(G)}$, where

$$\begin{cases} \|u\|^2_{\dot{W}_2^l(G)} = \displaystyle\sum_{|\alpha|=l} \|D^\alpha u\|^2_{L_2(G)} \quad \text{if } l \text{ is an integer}, \\ \|u\|^2_{\dot{W}_2^l(G)} = \displaystyle\sum_{|\alpha|=[l]} \int_G \int_G \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2\{l\}}} \mathrm{d}x\mathrm{d}y \\ \quad \text{if } l \text{ is a non-integer}, \quad l = [l] + \{l\}, \ 0 < \{l\} < 1. \end{cases}$$

For $l < 0$, we define $W_2^l(\mathbf{R}^n)$ as follows [3]:

$$W_2^l(\mathbf{R}^n) = \left\{u \ \middle|\ \|u\|^2_{W_2^l(\mathbf{R}^n)} \equiv \int_{\mathbf{R}^n} (1+|\xi|^2)^l |\hat{u}(\xi)|^2 \,\mathrm{d}\xi < \infty\right\}.$$

We also define the following function spaces:

$$\overline{W}_2^m(G) = \left\{u(x) \ \middle|\ \|u\|^2_{\overline{W}_2^m(G)} \equiv \sup_{x\in G} |u(x)|^2 + \|u\|^2_{\dot{W}_2^{m-[m]}(G)} + \sum_{|\alpha|=1} \|D^\alpha u\|^2_{W_2^{m-1}(G)} < \infty\right\}$$

$(n = 2, 3, \ m \in \mathbf{R})$. Next we introduce [3]

$$W_2^{l,\frac{l}{2}}(G_T) \equiv W_2^{l,0}(G_T) \cap W_2^{0,\frac{l}{2}}(G_T)$$

$(G_T \equiv G \times (0,T))$, whose norms are defined by

$$\|u\|^2_{W_2^{l,\frac{l}{2}}(G_T)} = \|u\|^2_{L_2((0,T); W_2^l(G))} + \|u\|^2_{L_2(\Omega; W_2^{\frac{l}{2}}(0,T))}.$$

We also define function spaces $\widetilde{W}_2^{l,\frac{l}{2}}(G_T) = \left\{u \in W_2^{l,\frac{l}{2}}(G_T) \ \middle|\ \dfrac{\partial u}{\partial x_3} \in W_2^{l,\frac{l}{2}}(G_T)\right\}$, and for $m > 2$

$$\overline{W}_2^{m,\frac{m}{2}}(G_T) = \left\{u(x,t) \ \middle|\ \|u\|^2_{\overline{W}_2^{m,\frac{m}{2}}(\widetilde{\Omega}_T)} \equiv \sup_{G_T} |u(x,t)|^2 + \sup_{t\in(0,T)} \|u(\cdot,t)\|^2_{\dot{W}_2^{m-[m]}(G)}\right.$$

$$+\sum_{|\alpha|=1}\|D^\alpha u\|^2_{W_2^{m-1,\frac{m-1}{2}}(G_T)}+\left\|\frac{\partial u}{\partial t}\right\|^2_{W_2^{m-2,\frac{m-2}{2}}(G_T)}$$
$$\left.+\sup_{x\in G}\|u(x,\cdot)\|^2_{\dot{W}_2^{\frac{m-[m]}{2}}(0,T)}<\infty\right\}.$$

The n times product of a function space W is denoted by W^n. Norms of the vector and the product spaces are defined by the standard vector norm and the sum of the norms of each space, respectively.

4. Main result. Now, introduce notations $\mathcal{U}\equiv(\mathbf{u},\tilde{\theta},\tilde{S})^{\mathrm{T}}$, $\mathcal{U}_0\equiv(\mathbf{u}_0,\tilde{\theta}_0,\tilde{S}_0)^{\mathrm{T}}$, and denote $(\mathbf{v},w,\theta,S)$ after the coordinate transform by $(\mathbf{u},u_3,\tilde{\theta},\tilde{S})$. Extend $\mathcal{U}_0(y)=(\mathbf{u}_0,\tilde{\theta}_0,\tilde{S}_0)(y)$ into the half space $t>0$ preserving the regularity, which is denoted by $\overline{\mathcal{U}}_0(y,t)=(\bar{\mathbf{u}}_0,\bar{\tilde{\theta}}_0,\bar{\tilde{S}}_0)(y,t)$. Introduce a notation $\mathcal{U}'=(\mathbf{u}',\tilde{\theta}',\tilde{S}')^{\mathrm{T}}\equiv\mathcal{U}-\overline{\mathcal{U}}_0$. We consider:

$$(4.1)\quad\begin{cases}\dfrac{\partial\mathcal{U}'}{\partial t}-\mathcal{L}_{\mathcal{U},F}\mathcal{U}'=\mathcal{G}_{1,u_3,F}\mathcal{U}'+\mathcal{L}_{\mathcal{U},F}\overline{\mathcal{U}}_0\\ \quad-\dfrac{\partial\overline{\mathcal{U}}_0}{\partial t}\quad\text{in }\widetilde{\Omega}_T,\\ \mathcal{B}_1(\nabla_{F,3}\mathbf{u},\tilde{\varrho},\nabla_{F,3}\tilde{\varrho})\mathcal{U}'=\mathcal{G}_{2,F}\mathcal{U}'\quad\text{on }\widetilde{\Gamma}_T,\\ \mathcal{B}_2(\nabla_{F,3}\mathbf{u},\tilde{\varrho},\nabla_{F,3}\tilde{\varrho})\mathcal{U}'=\mathcal{G}_{3,F}\mathcal{U}'\quad\text{on }\widetilde{\Gamma}_{bT},\\ \mathcal{U}'|_{t=0}=(\mathbf{0},0,0)^{\mathrm{T}},\end{cases}$$

where $\tilde{\varrho}=\varrho(\tilde{p}^{(F)},\tilde{\theta},\tilde{S})$, and

$$\mathcal{L}_{\mathcal{U},F}\mathcal{U}'\equiv[L_{1,\mathcal{U},F}(\mathbf{u}-\bar{\mathbf{u}}_0),L_{2,\mathcal{U},F}(\tilde{\theta}-\bar{\tilde{\theta}}_0),$$
$$L_{3,\mathcal{U},F}(\tilde{S}-\bar{\tilde{S}}_0)]^{\mathrm{T}},$$
$$\mathcal{B}_1(\nabla_{F,3}\mathbf{u},\tilde{\varrho},\nabla_{F,3}\tilde{\varrho})\mathcal{U}'$$
$$\equiv[\mu_2\nabla_F\mathbf{u}'\cdot\mathbf{n}_F,\ \mu_4\nabla_F\tilde{\theta}'\cdot\mathbf{n}_F,\ \mu_6\nabla_F\tilde{S}'\cdot\mathbf{n}_F]^{\mathrm{T}},$$
$$\mathcal{B}_2(\nabla_{F,3}\mathbf{u},\tilde{\varrho},\nabla_{F,3}\tilde{\varrho})\mathcal{U}'$$
$$\equiv[\mu_2\nabla_F\mathbf{u}'\cdot\mathbf{n}_b,\ \mu_4\nabla_F\tilde{\theta}'\cdot\mathbf{n}_b,\ \mu_6\nabla_F\tilde{S}'\cdot\mathbf{n}_b]^{\mathrm{T}},$$
$$L_{i,\mathcal{U},F}\equiv\mu_{2i-1}\nabla_F^2+\mu_{2i}\left(a^{33}(F)\frac{\partial}{\partial y_3}\right)^2\ (i=1,2,3),$$
$$\mathbf{G}_{1,F}(\mathbf{u},u_3)\equiv-A_1\frac{\partial\mathbf{u}}{\partial y_3}-(\mathbf{u}\cdot\nabla_F)\mathbf{u}-fA\mathbf{u}$$
$$-u_3a^{33}(F)\frac{\partial\mathbf{u}}{\partial y_3}-\frac{1}{\varrho_0}\nabla_F\tilde{p}^{(F)},$$
$$G_{2,F}(\mathbf{u},u_3)\equiv-A_1\frac{\partial}{\partial y_3}-(\mathbf{u}\cdot\nabla_F)-u_3a^{33}(F)\frac{\partial}{\partial y_3},$$
$$\mathcal{G}_{1,u_3,F}\mathcal{U}'\equiv[\mathbf{G}_{1,F},\ G_{2,F}(\mathbf{u},u_3)\tilde{\theta},\ G_{2,F}(\mathbf{u},u_3)\tilde{S}]^{\mathrm{T}},$$
$$\mathcal{G}_{2,F}\mathcal{U}'\equiv[\tilde{\tau}_1^{(F)}+\mu_2\nabla_F\bar{\mathbf{u}}_0\cdot\mathbf{n}_F,\tilde{\tau}_2^{(F)}+\mu_4\nabla_F\bar{\tilde{\theta}}_0\cdot\mathbf{n}_F,$$
$$\tilde{g}_1^{(F)}\tilde{S}+\mu_6\nabla_F\bar{\tilde{S}}_0\cdot\mathbf{n}_F]^{\mathrm{T}},$$
$$\mathcal{G}_{3,F}\mathcal{U}'\equiv[\mu_2\nabla_F\bar{\mathbf{u}}_0\cdot\mathbf{n}_b,\ \mu_4\nabla_F\bar{\tilde{\theta}}_0\cdot\mathbf{n}_b,\ \mu_6\nabla_F\bar{\tilde{S}}_0\cdot\mathbf{n}_b]^{\mathrm{T}},$$
$$\mathcal{L}_{\mathcal{U},F}\overline{\mathcal{U}}_0\equiv[L_{1,\mathcal{U},F}\bar{\mathbf{u}}_0,\ L_{2,\mathcal{U},F}\bar{\tilde{\theta}}_0,\ L_{3,\mathcal{U},F}\bar{\tilde{S}}_0]^{\mathrm{T}},$$
$$\mu_i=\mu_i(\nabla_{F,3}\mathbf{u},\tilde{\varrho},\nabla_{F,3}\tilde{\varrho})\ (i=2,4,6).$$

The problem for u_3 is deduced from (2.1) and (2.4):

$$(4.2)\quad\begin{cases}a^{33}(F)\dfrac{\partial u_3}{\partial y_3}=-\nabla_F\cdot\mathbf{u}\quad\text{in }\widetilde{\Omega}_T,\\ u_3=-\mathbf{u}\cdot\nabla b\quad\text{on }\widetilde{\Gamma}_b.\end{cases}$$

For $F'\equiv F-F_0$, taking the horizontal divergence of $(4.1)_2$, and adding it with (2.2) yields:

$$(4.3)\quad\begin{cases}\dfrac{\partial F'}{\partial t}-\mathcal{L}_{4,\mathcal{U},F}F'=\mathcal{L}_{4,\mathcal{U},F}F_0-\mathbf{u}\cdot\nabla F\\ \quad+u_3+\displaystyle\sum_{i=1}^2\frac{\partial}{\partial y_i}\left[\frac{a^{33}(F)\mu_2}{\sqrt{1+|\nabla F|^2}}\frac{\partial u_i}{\partial y_3}\right]\\ \quad-\nabla\cdot\tilde{\tau}_1^{(F)}|_{y_3=F_0}\quad\text{in }\mathbf{R}_T^2,\\ F'|_{t=0}=0\quad\text{on }\mathbf{R}^2,\end{cases}$$

where

$$\mathcal{L}_{4,\mathcal{U},F}F\equiv\sum_{i=1}^2\frac{\mu_2(\nabla_{F,3}\mathbf{u},\tilde{\varrho},\nabla_{F,3}\tilde{\varrho})}{\sqrt{1+|\nabla F|^2}}$$
$$\times\left(\nabla u_i+\mathbf{a}^3\frac{\partial u_i}{\partial y_3}\right)\cdot\nabla\frac{\partial F}{\partial y_i}.$$

Now, define $p_u(x)$ by

$$p_u(x)=p_0(x',F_0(x'),0)$$
$$+g\int_{F_0(x')}^{x_3}\varrho((p_u,\theta_0,S_0)(x',z_3,0))\,\mathrm{d}z_3,$$

We also define $\varrho_u\equiv\varrho(p_u,\theta_0,S_0)$ and

$$\mathcal{R}_0\equiv\mathcal{R}\left(\frac{\partial\mathbf{v}_0}{\partial x_3},\varrho_u,\frac{\partial\varrho_u}{\partial x_3}\right).$$

The following is the main result:

Theorem 4.1. *Let $l\in(1/2,1)$, and T be an arbitrary positive number. Assume that*

(i) $(\mathbf{v}_0,\theta_0,S_0)\in(W_2^{2+l}(\mathbf{R}^3)\bigcap W_2^{s_1}(\mathbf{R}^3))\times$ $(\overline{W}_2^{2+l}(\mathbf{R}^3)\bigcap W_2^{s_2}(\mathbf{R}^3))\times$ $(\overline{W}_2^{2+l}(\mathbf{R}^3)\bigcap W_2^{s_3}(\mathbf{R}^3))$ *for some* $s_i<0$ $(i=1,2,3)$,

(ii) $0<c_u\le(1+\alpha_i\mathcal{R}_0)\left|\dfrac{\partial\mathbf{v}_0}{\partial x_3}\right|$ $(i=2,4,6)$,

$F_0\in W_2^{\frac{5}{2}+l}(\mathbf{R}^2)$, $\underline{\theta}_0\le\theta_0(x)<\infty$, *and* $0<\underline{S}_0\le S_0(x)<\infty$ *with positive constants* c_u, $\underline{\theta}_0$ *and* $\underline{S}_0$, *respectively;*

(iii) $b \in \overline{W}_2^{\frac{5}{2}+l}(\mathbf{R}^2)$ *and* $F_0(x') - b(x') > c_0 > 0$ *on* $\mathbf{R}^2$ *with a positive constant* c_0*;*

(iv) $\tau_1,\ \tau_2 \in W_2^{2+l,\frac{2+l}{2}}(\mathbf{R}_T^3)$, $g_1 \in W_2^{2+l,\frac{2+l}{2}}(\mathbf{R}_T^3)$, $p_0 \in \overline{W}_2^{3+l,\frac{3+l}{2}}(\mathbf{R}_T^3)$, *and the matrix* $\mathbf{A}_{\mathbf{v}_0} \equiv \left(\dfrac{\partial v_{0i}}{\partial x_j}\right)_{i,j=1,2}$ *is positive definite;*

(v) *For* $l/2 < \beta < 1 + l/2$, $\varrho \in C^{4+\beta}(\mathcal{G})$ *on* $\mathcal{G} \equiv \{x \in \mathbf{R}^3 | x_1 > c_1, x_2 > \underline{\theta}_0, x_3 > \underline{S}_0\}$, $\inf\limits_{x\in\mathcal{G}} \varrho(x) \geq c_1 > 0$ *and* $\sup\limits_{x\in\mathcal{G}} |D^\alpha \varrho(x)| \leq M$ *for* $|\alpha| \leq 4$.

Moreover, compatibility conditions up to the order 1 are satisfied. Then, there exists $T^* \in (0, T]$ *such that* (4.1)–(4.3) *has a unique solution* $(\mathcal{U}', u_3, F') \in \mathcal{W}'(T^*) \times \widetilde{W}_2^{2+l,\frac{2+l}{2}}(\widetilde{\Omega}_{T^*}) \times W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_{T^*}^2)$ *satisfying* $0 < \frac{\underline{\theta}_0}{2} \leq \tilde{\theta}$ *and* $0 < \frac{\underline{S}_0}{2} \leq \tilde{S}$ *on* $\widetilde{\Omega}_{T^*}$, *where* $\mathcal{W}'(T^*) \equiv (W_2^{3+l,\frac{3+l}{2}}(\widetilde{\Omega}_{T^*}))^3$.

5. Auxiliary lemmas. We introduce some lemmas used in the proof of Theorem 4.1, whose proofs will be provided in a full paper published later. Hereafter, C's stand for positive constants depending on $\|b\|_{\overline{W}_2^{\frac{5}{2}+l}(\mathbf{R}^2)}$, $\|F_0\|_{W_2^{\frac{5}{2}+l}(\mathbf{R}^2)}$ and $\|p_0\|_{\overline{W}_2^{3+l,\frac{3+l}{2}}(\mathbf{R}_T^3)}$, and $\phi(\cdot)$'s for increasing positive functions of their arguments which include constants depending on the same amounts as C's. We assume the following assumption:

A number $T > 0$ and a function F_* satisfy $F_* \in W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)$; $F_*(y', t) - b(y') > c_0$ for $t \in [0, T]$ and $F_*(y', 0) = F_0(y')$. We shall call this assumtion as assumption (A_{F_*}). The transform (2.6) is executed with F replaced by F_*, and by $\widetilde{\Omega}_T$, we denote the region mapped from $\bigcup\limits_{t>0} \Omega(t) \times \{t\}$ with this transform. $b \in \overline{W}_2^{\frac{5}{2}+l}(\mathbf{R}^2)$, $F_0 \in W_2^{\frac{5}{2}+l}(\mathbf{R}^2)$, $F_0(y') - b(y') > c_0$. Hereafter, D^α stands for the usual multi-index of the derivative. In addition, we use notations

$$D_{i,j,k}^3 = \frac{\partial^3}{\partial x_i \partial x_j \partial x_k},$$

$$\widetilde{D}^\gamma[f](y,t) \equiv \sum_{|\alpha|=\gamma}\left(|D^\alpha f(y,t)| + \int_{b(y')}^{F_0(y')} |D^\alpha f(y', z_3, t)|\, \mathrm{d}z_3\right),$$

for simplicity.

Lemma 5.1. *For* $T > 0$ *and* F_* *satisfying the assumption* (A_{F_*})*, let us assume the same conditions as in* Theorem 4.1. *In addition, let us define* $\mathcal{M}_* \equiv (\tilde{\theta}_*, \tilde{S}_*) \in \mathcal{W}_{\mathcal{M}}(T) \equiv (\overline{W}_2^{3+l,\frac{3+l}{2}}(\widetilde{\Omega}_T))^2$, $(\tilde{\theta}_*, \tilde{S}_*)|_{t=0} = (\tilde{\theta}_0, \tilde{S}_0) \in (\overline{W}_2^{2+l}(\mathbf{R}^3))^2$, $\mathcal{N}_* \equiv (\mathcal{M}_*, F_*)$, $\mathcal{V}_* \equiv (\tilde{p}^{(F_*)}, \mathcal{M}_*)$, *where*

$$\tilde{p}^{(F_*)}(y,t) = \tilde{p}_0^{(F_*)}(y', F_0(y'), t) + \frac{g}{a^{33}(F_*)(y',t)} \times \int_{F_0(y')}^{y_3} \varrho((\tilde{p}^{(F_*)}, \tilde{\theta}_*, \tilde{S}_*)(y', z_3, t))\, \mathrm{d}z_3.$$

Then, we have:

$$\begin{aligned}
|D_{i,j,k}^3 \mathcal{V}_*(y,t)| &\leq \phi(\|\mathcal{N}_*\|_{\mathcal{W}_{\mathcal{N}}(t)}) \\
&\times \left\{\sum_{|\alpha=2|}^{3} |D^\alpha p_0(y,t)| + \left|D_{i,j,k}^3 F_0(y')\right| + \widetilde{D}^2[\mathcal{N}_*](y,t) \right.\\
&\left. + \int_{b(y')}^{F_0(y')} \left|D_{i,j,k}^3 \mathcal{M}_*(y', z_3, t)\right| \mathrm{d}z_3 + 1\right\} \\
&\times \left\{1 + \exp\left(C\left(1 + \|F_*\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\right)\right)\right\} \\
&+ D_{i,j,k}^3 \mathcal{M}_*(y,t) \quad (i,j,k = 1,2),\ \forall (y,t) \in \widetilde{\Omega}_T,
\end{aligned}$$

where

$$\mathcal{W}_{\mathcal{N}}(t) \equiv (\overline{W}_2^{3+l,\frac{3+l}{2}}(\widetilde{\Omega}_t))^2 \times W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2).$$

Next, assume $(\tilde{\theta}_{(i)}, \tilde{S}_{(i)}, F_{(i)})$ $(i = 1, 2)$ satisfy

$$(\tilde{\theta}_{(i)}, \tilde{S}_{(i)}, F_{(i)})|_{t=0} = (\tilde{\theta}_0(y), \tilde{S}_0(y), F_0(y')),$$
$$(\tilde{\theta}_{(i)}, \tilde{S}_{(i)}) = (\bar{\tilde{\theta}}_0, \bar{\tilde{S}}_0) + (\tilde{\theta}'_{(i)}, \tilde{S}'_{(i)}),$$
$$(\tilde{\theta}'_{(i)}, \tilde{S}'_{(i)}) \in (W_2^{3+l,\frac{3+l}{2}}(\widetilde{\Omega}_T))^2,$$
$$(\tilde{\theta}_{(i)}, \tilde{S}_{(i)}, F_{(i)}) \in \mathcal{W}_{\mathcal{N}}(T)$$

and $\tilde{p}_{(i)}$ satisfies

$$\tilde{p}_{(i)}(y,t) = \tilde{p}_0^{(F_{(i)})}(y', F_0(y'), t) + \frac{g}{a^{33}(F_{(i)})(y',t)} \times \int_{y_3}^{F_0(y')} \varrho((\tilde{p}_{(i)}, \tilde{\theta}_{(i)}, \tilde{S}_{(i)})(y', z_3, t))\, \mathrm{d}z_3 \ (i = 1, 2).$$

Introducing notations $\mathcal{V}_{(i)} \equiv (\tilde{p}_{(i)}, \tilde{\theta}_{(i)}, \tilde{S}_{(i)})$ $(i = 1, 2)$, we then estimate $\widetilde{\mathcal{V}} \equiv \mathcal{V}_{(2)} - \mathcal{V}_{(1)} \equiv (\tilde{\tilde{p}}, \tilde{\tilde{\theta}}, \tilde{\tilde{S}})$.

Lemma 5.2. *Let* $T > 0$ *satisfy* $F_{(i)}(y', t) - b(y') > c_0$ $(i = 1, 2)$ *for* $t \in [0, T]$. *For this* T*, under the same assumptions as in* Lemma 5.1*, the following estimates hold for* $t \in (0, T]$*:*

$$|D^3_{i,j,k}\widetilde{\mathcal{V}}(y,t)| \leq \phi\left(\sum_{i=1}^{2}\|\mathcal{N}_{(i)}\|_{\mathcal{W}_{\mathcal{N}}(T)}\right) \times \sum_{\gamma=2}^{3}\widetilde{D}^{\gamma}\widetilde{\mathcal{N}}(y,t) + C\|\widetilde{\mathcal{N}}\|_{\mathcal{W}_{\widetilde{\mathcal{N}}}(T)} \times \left\{\sum_{\gamma=2}^{3}|\widetilde{D}^{\gamma}[p_0](y,t)| + \sum_{k=1}^{2}\widetilde{D}^2[\mathcal{N}_{(k)}(y,t)] + \sum_{|\alpha|=3}(|D^{\alpha}F_0(y')| + |D^{\alpha}F_{(1)}(y',t)|) + 1\right\} + |D^3_{i,j,k}\widetilde{\mathcal{M}}(y,t)| \ (i,j,k=1,2),\ \forall (y,t)\in\widetilde{\Omega}_T,$$

with $\widetilde{\mathcal{W}}_{\mathcal{N}}(T) \equiv (W_2^{3+l,\frac{3+l}{2}}(\widetilde{\Omega}_T))^2 \times W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}^2_T)$.

6. Proof of Theorem 4.1. First, we construct the following successive approximation of the nonlinear problem for $\mathcal{U}'_{(m+1)}$ with $m \geq 0$:

$$\text{(6.1)}\quad \begin{cases} \dfrac{\partial \mathcal{U}'_{(m+1)}}{\partial t} - \mathcal{L}_{\mathcal{U}_{(m)},F_{(m)}}\mathcal{U}'_{(m+1)} \\ \quad = \mathcal{G}_{1,F_{(m)}}\mathcal{U}'_{(m)} + \mathcal{L}_{\mathcal{U}_{(m)},F_{(m)}}\overline{\mathcal{U}}_0 - \dfrac{\partial \overline{\mathcal{U}}_0}{\partial t} \\ \quad \equiv \mathcal{E}_1^{(m)} \quad \text{in } \widetilde{\Omega}_T, \\ \mathcal{B}_1(\nabla_{F_{(m)},3}\mathbf{u}_{(m)}, \tilde{\varrho}_{(m)}, \nabla_{F_{(m)},3}\tilde{\varrho}_{(m)})\mathcal{U}'_{(m+1)} \\ \quad = \mathcal{G}_{2,F_{(m)}}\mathcal{U}'_{(m)} \equiv \mathcal{E}_2^{(m)} \quad \text{on } \widetilde{\Gamma}_T, \\ \mathcal{B}_2(\nabla_{F_{(m)},3}\mathbf{u}_{(m)}, \tilde{\varrho}_{(m)}, \nabla_{F_{(m)},3}\tilde{\varrho}_{(m)})\mathcal{U}'_{(m+1)} \\ \quad = \mathcal{G}_{3,F_{(m)}}\mathcal{U}'_{(m)} \equiv \mathcal{E}_3^{(m)} \quad \text{on } \widetilde{\Gamma}_{bT}, \\ \mathcal{U}'_{(m+1)}|_{t=0} = (\mathbf{0},0,0) \quad \text{on } \widetilde{\Omega}. \end{cases}$$

For $m = 0$, we define

$$(\mathcal{U}'_{(m)}, u_{3(m)}, F'_{(m)}, \tilde{\varrho}_{(m)}) \equiv (\mathbf{0},0,0,0,0,\varrho(\tilde{p}_0,\bar{\tilde{\theta}}_0,\bar{\tilde{S}}_0)),$$
$$(\mathcal{U}_{(m)}, F_{(m)}) = (\mathcal{U}_0, F_0) = (\bar{\mathbf{u}}_0, \bar{\theta}_0, \bar{S}_0, F_0),$$
$$\mathcal{R}_{(m)} = g(\tilde{\varrho}_{(m)}a^{33}(F_0))^{-1}\frac{\partial \tilde{\varrho}_{(m)}}{\partial y_3}\left|\frac{\partial \bar{\mathbf{u}}_0}{\partial y_3}\right|^{-2},$$

and let us $\tilde{p}_{(m)}$ and $\tilde{\varrho}_{(m)} \equiv \varrho((\tilde{p}_{(m)}, \tilde{\theta}_{(m)}, \tilde{S}_{(m)})(y,t)) \equiv \varrho(\mathcal{V}_{(m)})$ satisfy for $m \geq 0$

$$\begin{cases} a^{33}_{(m)}\dfrac{\partial \tilde{p}_{(m)}}{\partial y_3} = -g\tilde{\varrho}_{(m)} \quad \text{in } \widetilde{\Omega}_T, \\ \tilde{p}_{(m)}|_{y_3=F_0(y')} = \tilde{p}_0(y', F_0(y'), t) \quad \text{on } \widetilde{\Gamma}_{bT}, \end{cases}$$

where $a^{33}_{(m)} = a^{33}(F_{(m)})$. We also consider:

$$\text{(6.2)}\quad \begin{cases} a^{33}_{(m)}\dfrac{\partial u_{3(m+1)}}{\partial y_3} = -\nabla_{F_{(m)}}\cdot \mathbf{u}_{(m)} \quad \text{in } \widetilde{\Omega}_T, \\ u_{3(m+1)} = -\widetilde{\mathbf{u}}_{(m+1)}\cdot\nabla b \quad \text{on } \widetilde{\Gamma}_{bT}, \end{cases}$$

$$\text{(6.3)}\quad \begin{cases} \dfrac{\partial F'_{(m+1)}}{\partial t} - \mathcal{L}_{4,\mathcal{U}_{(m)},F_{(m)}}F'_{(m+1)} \\ \quad = \mathcal{L}_{4,\mathcal{U}_{(m)},F_{(m)}}F_0 - \mathbf{u}_{(m)}\cdot F_{(m)} + u_{3(m+1)} \\ \qquad + \displaystyle\sum_{i=1}^{2}\frac{\partial}{\partial y_i}\left[\frac{a^{33}_{(m)}\mu_{2(m)}}{\sqrt{1+|\nabla F_{(m)}|^2}}\frac{\partial u_{i(m)}}{\partial y_3}\right] \\ \qquad -\nabla\cdot\tau_1^{(F_{(m)})} \equiv \mathcal{E}_4^{(m)} \text{ in } \mathbf{R}^2_T, \\ F'_{(m+1)}|_{t=0} = 0 \quad \text{on } \mathbf{R}^2. \end{cases}$$

Here $\mu_{2(m)} = \mu_2(\nabla_{F_{(m)},3}\mathbf{u}_{(m)}, \tilde{\varrho}_{(m)}, \nabla_{F_{(m)},3}\tilde{\varrho}_{(m)})$. The unique solvability of (6.1)–(6.3) is guaranteed by the result of the linear problem. Making use of the interpolation and Young's inequalities and the estimate of the solution to the linear problem, we show the well-definedness of the successive sequence.

Hereafter $\mathcal{R}_{(m)} \equiv \dfrac{g\tilde{\varrho}^{-1}_{(m)}}{a^{33}_{(m)}}\dfrac{\partial \tilde{\varrho}_{(m)}}{\partial y_3}\left|\dfrac{\partial \mathbf{u}_{(m)}}{\partial y_3}\right|^{-2}$ $(m \geq 1)$.

We also introduce following function spaces:

$$\mathcal{W}_1(T) \equiv (W_2^{1+l,\frac{1+l}{2}}(\widetilde{\Omega}_T))^3,$$
$$\mathcal{W}_2(T) \equiv (W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\widetilde{\Gamma}_T))^3$$
$$\mathcal{W}_3(T) \equiv (W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\widetilde{\Gamma}_{bT}))^3.$$

Lemma 6.1. *Assume* $F_0 \in W_2^{\frac{5}{2}+l}(\mathbf{R}^2)$, $\mathcal{U}_0 \equiv (\mathbf{u}_0, \tilde{\theta}_0, \tilde{S}_0) \in \mathcal{W}_0 \equiv W_2^{2+l}(\widetilde{\Omega}) \times \overline{W}_2^{2+l}(\widetilde{\Omega}) \times \overline{W}_2^{2+l}(\widetilde{\Omega})$, $(1+\alpha_i\mathcal{R}_0)\left|\dfrac{\partial \mathbf{v}_0}{\partial x_3}\right| \geq c_u > 0$ $(i = 2,4,6)$ *and there exists* $T_{60} > 0$ *satisfying* $F_{(m)}(y',t) - b(y') > c_0 > 0$ *and* $(1+\alpha_i\mathcal{R}_{(m)})\left|\dfrac{\partial \mathbf{u}_{(m)}}{\partial y_3}\right| > c_u > 0$ $(i = 1,2,3)$ *for* $t \in (0, T_{60}]$. *Then, following estimates hold for arbitrary small* $\epsilon > 0$ *and* $t \in (0, T_{60}]$*:*

$$\sum_{i=1}^{2}\|\mathcal{E}_i^{(m)}\|_{\mathcal{W}_1(t)} \leq (\epsilon + C_\epsilon t)(1 + \|g_1\|_{W_2^{2+l,\frac{2+l}{2}}(\mathbf{R}^3_t)}) \times \phi(\|F_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}^2_t)})\|\mathcal{U}'_{(m)}\|_{\mathcal{W}'(t)} + \sum_{i=1}^{2}\|\tau_i\|_{W_2^{2+l,\frac{2+l}{2}}(\mathbf{R}^3_t)} + \|\mathcal{U}_0\|_{\mathcal{W}_0},$$

$$\sum_{i=3}^{4}\|\mathcal{E}_i^{(m)}\|_{\mathcal{W}_3(t)} \leq (\epsilon + C_\epsilon t)(1 + \|b\|_{\overline{W}_2^{\frac{5}{2}+l}(\mathbf{R}^2)}) \times \|\mathcal{U}'_{(m)}\|_{\mathcal{W}'(t)} + \|\mathcal{U}_0\|_{\mathcal{W}_0} + \|F_0\|_{W_2^{\frac{5}{2}+l}(\mathbf{R}^2)},$$

where C_ϵ *is a positive constant depending on* ϵ.

This lemma is proved with the aid of Lemma 5.1. Introduce notations $E'_m(t) \equiv \|\mathcal{U}'_{(m)}\|_{\mathcal{W}'(t)}$ and $E_m(t) \equiv \|(\mathcal{U}'_{(m)}, u_{3(m)}, F_{(m)})\|_{\mathcal{W}(t)}$.

Take T_{61} such that $F_{(m)}(y', t) - b(y') > c_0$ and

$$(1 + \alpha_i \mathcal{R}_{(m)})\left|\frac{\partial \mathbf{u}_{(m)}}{\partial y_3}\right| > c_u > 0$$

hold for $t \in (0, T_{61}]$. The estimate of the solution to the linear problem and Lemma 6.1 yields:

$$\begin{aligned} &E'_{(m+1)}(t) \le C[(\epsilon + C_\epsilon t)\{\phi_{61}(E_{(m)}(t)) \\ &\quad + \phi_{62}(E_{(m)}(t))E'_{(m)}(t)\} + 1], \\ &\|u_{3(m+1)}\|_{\widetilde{W}_2^{3+l,\frac{3+l}{2}}(\widetilde{\Omega}_t)} + \|F_{(m+1)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \\ &\quad \le \left\{\phi_{63}(\|F'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}) + C\right\} \\ &\quad \times \|\mathbf{u}_{(m+1)}\|_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)} \\ &\quad + C\{(\epsilon + C_\epsilon t)\phi_{64}(E_{(m)}(t)) + 1\}. \end{aligned}$$

Here $\phi_{6i}(\cdot)$ $(i = 1, 2, 3, 4)$ are homogeneous polynomials. From these, we arrive at:

$$\begin{aligned} E_{(m+1)}(t) \le C(t)[(\epsilon + C_\epsilon t)\{&\phi_{61}(E_{(m)}(T)) \\ &+ \phi_{62}(E_{(m)}(T))E'_{(m)}(T) \\ &+ \phi_{64}(E_{(m)}(T))\} + 1] \end{aligned} \tag{6.4}$$

with some $C(t) \ge 0$ depending on t monotonically and increasingly. From this, we obtain $E_{(m+1)}(T_{61}) < M$ from the assumption $E_{(m)}(T_{61}) < M$ by taking ϵ and T_{61} small enough. With the aid of the following lemma, it is shown that T_{61} does not depend on m.

Lemma 6.2. *Let $M > 0$ be provided as above. Then, there exists a constant $C_{60} > 0$ independent of $E_{(m)}$ such that*

$$(1 + \alpha_i \mathcal{R}_{(m)}(y,t))\left|\frac{\partial \mathbf{u}_{(m)}}{\partial y_3}(y,t)\right| > \frac{c_u}{2} \quad (i = 2, 4, 6)$$

hold for $t \in \left(0, \dfrac{c_u}{C_{60}M}\right)$.

Next, we prove the convergence of the sequence. Subtract (6.1)–(6.3) with m replaced by $m-1$ from itself. Denoting

$$\widetilde{\mathcal{U}}'_{(m)} \equiv \mathcal{U}'_{(m+1)} - \mathcal{U}'_{(m)}, \ \tilde{\mathbf{u}}_{(m)} \equiv \mathbf{u}_{(m+1)} - \mathbf{u}_{(m)},$$
$$\tilde{u}_{3(m)} \equiv u_{3(m+1)} - u_{3(m)}, \ \widetilde{F}'_{(m)} \equiv F'_{(m+1)} - F'_{(m)},$$

we consider the similar problem for these variables.

Lemma 6.3. *Under the same assumptions as in* Lemma 6.1*, following estimate holds for ϵ, $t > 0$:*

$$\begin{aligned} &\sum_{i=1}^{3} \|\mathcal{E}_i^{(m)} - \mathcal{E}_i^{(m-1)}\|_{\mathcal{W}_i(t)} \\ &\quad + \|\mathcal{E}_4^{(m)} - \mathcal{E}_4^{(m-1)}\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \\ &\quad \le (\epsilon + C_\epsilon t)\phi\left(\sum_{i=m-1}^{m} E_i(t)\right)\widetilde{E}_m(t), \end{aligned}$$

where C_ϵ is a potisive constant depending on ϵ.

This lemma is proved with the aid of Lemma 5.2. From Lemma 6.3, we obtain

$$\widetilde{E}_{(m+1)}(t) \le r\widetilde{E}_{(m)}(t), \ r \in (0,1),$$

where $\widetilde{E}_m(t) \equiv \|(\widetilde{\mathcal{U}}'_{(m)}, \tilde{u}_{3(m)}, \widetilde{F}'_{(m)})\|_{\mathcal{W}(t)}$. Then we can verify that $\{(\widetilde{\mathcal{U}}'_{(m)}, \tilde{u}_{3(m)}, \widetilde{F}'_{(m)})\}_{m=0}^{\infty}$ is a Cauchy sequence in $\mathcal{W}(T_{62})$. Therefore the limit

$$(\widetilde{\mathcal{U}}', \tilde{u}_3, \widetilde{F}') \equiv \lim_{m \to \infty} (\widetilde{\mathcal{U}}'_{(m)}, \tilde{u}_{3(m)}, \widetilde{F}'_{(m)})$$

exists in $\mathcal{W}(T_{62})$, which is our desired solution. Finally we shall show that $0 < \underline{\theta}_0/2 \le \tilde{\theta}(y,t) < \infty$ and $0 < \underline{S}_0/2 \le \tilde{S}(y,t) < \infty$ hold by taking the time interval small enough again. This is achieved in the similar manner as in [3]. Uniqueness of the solution can be proved by virtue of an analogous inequality to (6.4). This completes the proof of the main theorem.

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