Remark on energy density of Brody curves

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Abstract: We introduce several definitions of energy density of Brody curves and show that they give the same value in an appropriate situation.

Key words: Brody curve; energy density.

1. Introduction. Let $z = x + y\sqrt{-1} \in \mathbf{C}$ be the standard coordinate of the complex plane $\mathbf{C}$. Let X be a compact Hermitian manifold with the Kähler form ω, and let $f : \mathbf{C} \to X$ be a holomorphic map. We define the spherical derivative $|df|(z) \geq 0$ by

$$f^*\omega = |df|^2 dxdy.$$

We call f a Brody curve (cf. Brody [1]) if it satisfies $|df|(z) \leq 1$ for all $z \in \mathbf{C}$. Let $\mathcal{M}(X)$ be the space of Brody curves in X. This is equipped with the compact-open topology, and it becomes a compact metrizable space (possibly infinite dimensional) with the following natural continuous $\mathbf{C}$-action:

$$\mathbf{C} \times \mathcal{M}(X) \to \mathcal{M}(X), \quad (a, f(z)) \mapsto f(z+a).$$

For $f \in \mathcal{M}(X)$, we define the energy density $\rho(f)$ (first introduced in [4]) by

$$\rho(f) := \lim_{R\to\infty}\left(\frac{1}{\pi R^2}\sup_{a\in\mathbf{C}}\int_{|z-a|<R}|df|^2 dxdy\right).$$

(This limit always exists by Lemma 2.4 in Section 2.) Let $\mathcal{N} \subset \mathcal{M}(X)$ be a $\mathbf{C}$-invariant closed subset. We define $\rho(\mathcal{N})$ as the supremum of $\rho(f)$ over all $f \in \mathcal{N}$. We sometimes denote $\rho(\mathcal{M}(X))$ by $\rho(X)$.

The idea of introducing $\rho(\mathcal{N})$ began in the paper [7]. ([7] uses a different definition.) It has a close relation to the mean dimension theory (introduced by Gromov [3]). The paper [4] proves

$$2(N+1)\rho(\mathbf{C}P^N) \leq \dim(\mathcal{M}(\mathbf{C}P^N) : \mathbf{C}) \leq 4N\rho(\mathbf{C}P^N).$$

Here $\mathbf{C}P^N$ is the projective space with the standard Fubini-Study metric, and $\dim(\mathcal{M}(\mathbf{C}P^N) : \mathbf{C})$ is the mean dimension of $\mathcal{M}(\mathbf{C}P^N)$. In particular

$$\dim(\mathcal{M}(\mathbf{C}P^1) : \mathbf{C}) = 4\rho(\mathbf{C}P^1).$$

The purpose of the present paper is to study variants of $\rho(\mathcal{N})$ and to show that they give the same value.

Let $T(r, f)$ be the Nevanlinna-Shimizu-Ahlfors characteristic function of $f \in \mathcal{M}(X)$:

$$T(r,f) := \int_1^r\left(\int_{|z|<t}|df|^2 dxdy\right)\frac{dt}{t} \quad (r \geq 1).$$

Since $|df| \leq 1$ we have $T(r,f) \leq \pi r^2/2$. We define $\rho_{\mathrm{NSA}}(f)$ and $\underline{\rho}_{\mathrm{NSA}}(f)$ by

$$\rho_{\mathrm{NSA}}(f) := \limsup_{r\to\infty}\frac{2}{\pi r^2}T(r,f),$$

$$\underline{\rho}_{\mathrm{NSA}}(f) := \liminf_{r\to\infty}\frac{2}{\pi r^2}T(r,f).$$

For a $\mathbf{C}$-invariant closed subset $\mathcal{N} \subset \mathcal{M}(X)$, let $\rho_{\mathrm{NSA}}(\mathcal{N})$ and $\underline{\rho}_{\mathrm{NSA}}(\mathcal{N})$ be the supremums of $\rho_{\mathrm{NSA}}(f)$ and $\underline{\rho}_{\mathrm{NSA}}(f)$ over $f \in \mathcal{N}$ respectively. It is easy to see $\underline{\rho}_{\mathrm{NSA}}(f) \leq \rho_{\mathrm{NSA}}(f) \leq \rho(f)$. Hence $\underline{\rho}_{\mathrm{NSA}}(\mathcal{N}) \leq \rho_{\mathrm{NSA}}(\mathcal{N}) \leq \rho(\mathcal{N})$.

The quantity $\rho_{\mathrm{NSA}}(\mathcal{M}(X))$ naturally appeared in the study of the upper bound on the mean dimension [6].

Example 1.1. Consider $\mathbf{Z}^2 = \{(x,y) \mid x, y \in \mathbf{Z}\} \subset \mathbf{C}$. Let a_n $(n \geq 1)$ be an increasing sequence of positive numbers which goes to infinity sufficiently fast. ($a_n = n^2$ will do.) Set

$$\Lambda := \mathbf{Z}^2 \cap \left(\bigcup_{n=1}^{\infty}\{z \in \mathbf{C} \mid |z - a_n| \leq n\}\right).$$

Let $c > 0$. We define a meromorphic function $f(z)$ by

$$f(z) := \sum_{\lambda\in\Lambda}\frac{1}{(cz-\lambda)^3}.$$

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We can choose c so that $f \in \mathcal{M}(\mathbf{C}P^1)$ and

$$\rho(f) > 0, \quad \rho_{\mathrm{NSA}}(f) = \underline{\rho}_{\mathrm{NSA}}(f) = 0.$$

For $f \in \mathcal{M}(X)$ we denote the closure of the $\mathbf{C}$-orbit of f by $\overline{\mathbf{C} \cdot f}$. Our main result is the following

Theorem 1.2. *For any $f \in \mathcal{M}(X)$ we have*

$$\rho(f) = \rho(\overline{\mathbf{C} \cdot f}) = \rho_{\mathrm{NSA}}(\overline{\mathbf{C} \cdot f}) = \underline{\rho}_{\mathrm{NSA}}(\overline{\mathbf{C} \cdot f}).$$

Hence for any $\mathbf{C}$-invariant closed subset $\mathcal{N} \subset \mathcal{M}(X)$

$$\rho(\mathcal{N}) = \rho_{\mathrm{NSA}}(\mathcal{N}) = \underline{\rho}_{\mathrm{NSA}}(\mathcal{N}).$$

The technique of the proof of Theorem 1.2 also gives the following

Theorem 1.3. *For any $\mathbf{C}$-invariant closed subset $\mathcal{N} \subset \mathcal{M}(X)$*

$$\rho(\mathcal{N}) = \lim_{R\to\infty}\left(\frac{1}{\pi R^2}\sup_{f\in\mathcal{N}}\int_{|z|<R}|df|^2 dxdy\right). \tag{1}$$

The proofs of these theorems will be given in Section 3. The essential ingredients of the proofs are the standard argument of normal family (i.e. the compactness of $\mathcal{M}(X)$) and a technical result given in Section 2.

2. Technical result. We fix a positive integer D throughout this section. (Later we will need only the case $D = 2$.)

We introduce one notation on Borel measures: Let μ be a Borel measure on $\mathbf{R}^D$, and let $a \in \mathbf{R}^D$. We define a Borel measure $a.\mu$ on $\mathbf{R}^D$ by $(a.\mu)(\Omega) := \mu(a + \Omega)$ where $\Omega \subset \mathbf{R}^D$ and $a + \Omega := \{a + x \mid x \in \Omega\} \subset \mathbf{R}^D$.

Let $\mathcal{M}$ be a set of Borel measures on $\mathbf{R}^D$ satisfying the following two conditions:

(a) For any $\mu \in \mathcal{M}$ and $a \in \mathbf{R}^D$ we have $a.\mu \in \mathcal{M}$.

(b) $\sup_{\mu\in\mathcal{M}} \mu([0,1]^D) < +\infty$.

Under the condition (a), the condition (b) is equivalent to the condition that for every bounded Borel subset $\Omega \subset \mathbf{R}^D$ we have $\sup_{\mu\in\mathcal{M}} \mu(\Omega) < +\infty$.

Example 2.1. Let $\varphi : \mathbf{R}^D \to [0,1]$ be a measurable function, and set

$$\mu(\Omega) := \int_\Omega \varphi d\mathrm{vol}, \quad (\Omega \subset \mathbf{R}^D).$$

Here $d$vol is the standard volume element of $\mathbf{R}^D$. Then the set $\{a.\mu \mid a \in \mathbf{R}^D\}$ satisfies the above two conditions.

For a Borel set $\Omega \subset \mathbf{R}^D$ we denote its Lebesgue measure by $|\Omega|$. For $r > 0$ and $a \in \mathbf{R}^D$ we set $B_r(a) := \{x \in \mathbf{R}^D \mid |x - a| \le r\}$. We denote $B_r(0)$ by B_r. We introduce the following two quantities:

$$\rho := \lim_{R\to\infty}\left(\frac{1}{|B_R|}\sup_{\mu\in\mathcal{M}}\mu(B_R)\right),$$

$$\tilde{\rho} := \lim_{r\to\infty}\left[\lim_{R\to\infty}\left\{\sup_{\mu\in\mathcal{M}}\left(\inf_{r\le t\le R}\frac{\mu(B_t)}{|B_t|}\right)\right\}\right].$$

The existence of the limit in the definition of ρ follows from Lemma 2.4 below (see the proof of Lemma 2.5). The quantity

$$\sup_{\mu\in\mathcal{M}}\left(\inf_{r\le t\le R}\frac{\mu(B_t)}{|B_t|}\right)$$

is a non-increasing function in R and a non-decreasing function in r. Hence the limits in the definition of $\tilde{\rho}$ exist.

The definition of $\tilde{\rho}$ looks complicated, but it is easy to see $\tilde{\rho} \le \rho$. The following result is the main technical tool for the proofs of Theorems 1.2 and 1.3.

Theorem 2.2. $\tilde{\rho} = \rho$.

This result might be known to some specialists in harmonic analysis or ergodic theory. But I could not find a literature containing this result.

We need two lemmas below. Lemma 2.3 is the well-known finite Vitali covering lemma (see e.g. Einsiedler-Ward [2, p. 40, Lemma 2.27]). Lemma 2.4 is a special case of Ornstein-Weiss's lemma. (This formulation is due to Gromov [3, p. 336]. The original argument was given in Ornstein-Weiss [5, Chapter I, Sections 2 and 3].)

Lemma 2.3. *Let $a_1, \dots, a_K \in \mathbf{R}^D$ and $r_1, \dots, r_K > 0$. Then we can choose $1 \le i(1) < \dots < i(k) \le K$ such that the balls $B_{r_{i(1)}}(a_{i(1)}), \dots, B_{r_{i(k)}}(a_{i(k)})$ are disjoint and*

$$\bigcup_{j=1}^K B_{r_j}(a_j) \subset \bigcup_{j=1}^k B_{3r_{i(j)}}(a_{i(j)}).$$

Before giving the statement of Lemma 2.4 we need to prepare some terminologies. Let $\Omega \subset \mathbf{R}^D$ and $r > 0$. We define $\partial_r\Omega$ as the set of points $x \in \mathbf{R}^D$ such that $B_r(x)$ has a non-empty intersection both with Ω and $\mathbf{R}^D \setminus \Omega$. A sequence of bounded Borel subsets $\{\Omega_n\}_{n\ge1}$ of $\mathbf{R}^D$ is called a Følner sequence if for all $r > 0$

$$|\partial_r\Omega_n|/|\Omega_n| \to 0 \quad (n \to \infty).$$

The sequence $\{B_n\}_{n\ge1}$ is a Følner sequence. The sequence $\{[0,n]^D\}_{n\ge1}$ is also.

Lemma 2.4. *Let h be a non-negative function on the set of bounded Borel subsets of $\mathbf{R}^D$ satisfying the following three conditions.*

(Monotonicity) If $\Omega_1 \subset \Omega_2$, then $h(\Omega_1) \le h(\Omega_2)$.
(Subadditivity) $h(\Omega_1 \cup \Omega_2) \le h(\Omega_1) + h(\Omega_2)$.
(Invariance) For any $a \in \mathbf{R}^D$ and any bounded Borel subset $\Omega \subset \mathbf{R}^D$, we have $h(a+\Omega) = h(\Omega)$.

Then for any Følner sequence Ω_n $(n \ge 1)$ in $\mathbf{R}^D$, the limit of the sequence

$$h(\Omega_n)/|\Omega_n| \quad (n \ge 1)$$

exists, and its value is independent of the choice of a Følner sequence.

The following is an immediate consequence of Lemma 2.4.

Lemma 2.5. *For any $\varepsilon > 0$ there exists $N = N(\varepsilon) > 0$ such that every bounded Borel subset $\Omega \subset \mathbf{R}^D$ with $|\partial_N \Omega|/|\Omega| < 1/N$ satisfies*

$$\left| \frac{\sup_{\mu \in \mathcal{M}} \mu(\Omega)}{|\Omega|} - \rho \right| < \varepsilon.$$

Proof. Set $h(\Omega) := \sup_{\mu \in \mathcal{M}} \mu(\Omega)$. This satisfies the three conditions in Lemma 2.4. If the above statement is false, then there exist $\varepsilon > 0$ and a sequence of bounded Borel subsets $\Omega_n \subset \mathbf{R}^D$ with $|\partial_n \Omega_n|/|\Omega_n| < 1/n$ satisfying

$$\left| \frac{h(\Omega_n)}{|\Omega_n|} - \rho \right| \ge \varepsilon. \tag{2}$$

But $\{\Omega_n\}_{n \ge 1}$ satisfies the definition of a Følner sequence. So by Lemma 2.4

$$\rho = \lim_{n \to \infty} h(\Omega_n)/|\Omega_n|.$$

This contradicts the above (2). □

Proof of Theorem 2.2. Assume $\tilde{\rho} < \rho - \delta$ for some $\delta > 0$. Set $\varepsilon := \delta/(2 \cdot 3^{D+1})$. Let $N = N(\varepsilon)$ be a positive number given by Lemma 2.5. We choose $r > 0$ sufficiently large so that every $t \ge r$ satisfies

$$\frac{|\partial_N B_t|}{|B_t|} < \frac{1}{3N}. \tag{3}$$

We fix $R > r$ so that

$$\sup_{\mu \in \mathcal{M}} \left(\inf_{r \le t \le R} \frac{\mu(B_t)}{|B_t|} \right) < \rho - \delta.$$

Let $L > R$ be a large number satisfying

$$|B_{L-R}| > \frac{|B_L|}{3}, \quad \left(\frac{1}{2} - \frac{1}{3^{D+1}} \right) |B_L| > |B_R|. \tag{4}$$

Fix an arbitrary $\mu \in \mathcal{M}$. For each $a \in \mathbf{R}^D$ there is $t = t(a) \in [r, R]$ such that

$$\frac{\mu(B_t(a))}{|B_t|} = \frac{(a.\mu)(B_t)}{|B_t|} < \rho - \delta. \tag{5}$$

By the finite Vitali covering lemma (Lemma 2.3), we can choose $a_1, \dots, a_K \in B_{L-R}$ (set $t_i := t(a_i)$) such that $B_{t_i}(a_i) \cap B_{t_j}(a_j) = \emptyset$ $(i \ne j)$ and

$$B_{L-R} \subset \bigcup_{i=1}^K B_{3t_i}(a_i).$$

By the first condition of (4)

$$3^{-D-1}|B_L| < \sum_{i=1}^K |B_{t_i}(a_i)|.$$

Then we can choose (using the second condition of (4)) $1 \le J \le K$ such that

$$3^{-D-1}|B_L| < \sum_{i=1}^J |B_{t_i}(a_i)| \le \frac{|B_L|}{2}. \tag{6}$$

By (5)

$$\mu\left(\bigcup_{i=1}^J B_{t_i}(a_i) \right) < (\rho - \delta) \left| \bigcup_{i=1}^J B_{t_i}(a_i) \right|. \tag{7}$$

Set $\Omega := B_L \setminus \bigcup_{i=1}^J B_{t_i}(a_i)$. $|\Omega| \ge |B_L|/2$. Since $\partial_N \Omega \subset \partial_N B_L \cup \bigcup_{i=1}^J \partial_N B_{t_i}(a_i)$,

$$\begin{aligned} |\partial_N \Omega| &\le |\partial_N B_L| + \sum_{i=1}^J |\partial_N B_{t_i}(a_i)| \\ &< \frac{1}{3N} \left(|B_L| + \sum_{i=1}^J |B_{t_i}(a_i)| \right) \quad \text{(by (3))} \\ &\le \frac{|B_L|}{2N} \le \frac{|\Omega|}{N} \quad \text{(by (6))}. \end{aligned}$$

Hence by Lemma 2.5

$$\frac{\mu(\Omega)}{|\Omega|} < \rho + \varepsilon.$$

So by (7)

$$\begin{aligned} \mu(B_L) &= \mu(\Omega) + \mu\left(\bigcup_{i=1}^J B_{t_i}(a_i) \right) \\ &< (\rho + \varepsilon)|\Omega| + (\rho - \delta) \left| \bigcup_{i=1}^J B_{t_i}(a_i) \right| \\ &= \rho |B_L| + \underbrace{\left(\varepsilon |\Omega| - \delta \left| \bigcup_{i=1}^J B_{t_i}(a_i) \right| \right)}_{A}. \end{aligned}$$

By (6) and $\varepsilon = \delta/(2 \cdot 3^{D+1})$,

$$A < \varepsilon |B_L| - \delta \cdot 3^{-D-1} |B_L| = -\varepsilon |B_L|.$$

Thus

$$\frac{\mu(B_L)}{|B_L|} < \rho - \varepsilon.$$

Since $\mu \in \mathcal{M}$ is arbitrary,

$$\frac{1}{|B_L|}\sup_{\mu\in\mathcal{M}} \mu(B_L) \le \rho - \varepsilon.$$

We can let $L \to +\infty$. Hence $\rho \le \rho - \varepsilon$. This is a contradiction. □

Remark 2.6. In the above proof we have not used the complete additivity of measures $\mu \in \mathcal{M}$. We needed only the monotonicity and subadditivity (two conditions given in Lemma 2.4) of $\mu \in \mathcal{M}$. So Theorem 2.2 can be also applied to a set of monotone, subadditive, non-negative functions on the set of bounded Borel subsets of $\mathbf{R}^D$ satisfying the conditions (a) and (b) in the beginning of this section. This generalization is not used in this paper. But it might become useful in some future.

Applying Theorem 2.2 to Example 2.1, we get the following corollary:

Corollary 2.7. *Let* $\varphi : \mathbf{R}^D \to [0,1]$ *be a measurable function. Then*

$$\lim_{R\to\infty}\left(\frac{1}{|B_R|}\sup_{a\in\mathbf{R}^D}\int_{B_R(a)}\varphi d\mathrm{vol}\right) = \lim_{r\to\infty}\left[\lim_{R\to\infty}\left\{\sup_{a\in\mathbf{R}^D}\left(\inf_{r\le t\le R}\frac{\int_{B_t(a)}\varphi d\mathrm{vol}}{|B_t|}\right)\right\}\right].$$

3. Proofs of Theorems 1.2 and 1.3. Let $f : \mathbf{C} \to X$ be a Brody curve. We first prove Theorem 1.2.

Step 1. $\rho(f) = \rho(\overline{\mathbf{C}\cdot f})$.

Proof. It is enough to prove that $\rho(g) \le \rho(f)$ for all $g \in \overline{\mathbf{C}\cdot f}$. Take a sequence $\{a_n\}_{n\ge1} \subset \mathbf{C}$ such that $f(z+a_n)$ converges to $g(z)$ uniformly over every compact subset of $\mathbf{C}$. Let $\varepsilon > 0$. For any $R > 0$ and $b \in \mathbf{C}$ there exists $n_0 > 0$ such that for $n \ge n_0$

$$\left||df|^2(z+a_n) - |dg|^2(z)\right| < \varepsilon \quad (|z-b| < R).$$

Hence for $n \ge n_0$

$$\begin{aligned}\frac{1}{\pi R^2}&\int_{|z-b|<R}|dg|^2 dxdy \\ &\le \frac{1}{\pi R^2}\int_{|z-a_n-b|<R}|df|^2(z)dxdy + \varepsilon \\ &\le \frac{1}{\pi R^2}\sup_{a\in\mathbf{C}}\int_{|z-a|<R}|df|^2 dxdy + \varepsilon.\end{aligned}$$

Taking the supremum with respect to b and $R \to \infty$, we get $\rho(g) \le \rho(f) + \varepsilon$. Let $\varepsilon \to 0$. We get $\rho(g) \le \rho(f)$. □

Step 2. $\rho(f) = \underline{\rho}_{\mathrm{NSA}}(\overline{\mathbf{C}\cdot f}) = \rho_{\mathrm{NSA}}(\overline{\mathbf{C}\cdot f})$. (This completes the proof of Theorem 1.2.)

Proof. From Step 1, we get $\underline{\rho}_{\mathrm{NSA}}(\overline{\mathbf{C}\cdot f}) \le \rho_{\mathrm{NSA}}(\overline{\mathbf{C}\cdot f}) \le \rho(\overline{\mathbf{C}\cdot f}) = \rho(f)$. So it is enough to prove $\underline{\rho}_{\mathrm{NSA}}(\overline{\mathbf{C}\cdot f}) \ge \rho(f)$. By Corollary 2.7 $\rho(f)$ is equal to

$$\lim_{r\to\infty}\left[\lim_{R\to\infty}\left\{\sup_{a\in\mathbf{C}}\left(\inf_{r\le t\le R}\frac{\int_{B_t(a)}|df|^2 dxdy}{\pi t^2}\right)\right\}\right].$$

Let $\varepsilon > 0$ and fix $r = r(\varepsilon) > 1$ satisfying

$$\lim_{R\to\infty}\left\{\sup_{a\in\mathbf{C}}\left(\inf_{r\le t\le R}\frac{\int_{B_t(a)}|df|^2 dxdy}{\pi t^2}\right)\right\} > \rho(f) - \varepsilon.$$

Then for any $R > r$ there exists $a(R) \in \mathbf{C}$ such that

$$\inf_{r\le t\le R}\frac{1}{\pi t^2}\int_{B_t(a(R))}|df|^2(z)dxdy > \rho(f) - \varepsilon.$$

Since $\mathcal{M}(X)$ is compact, we can take a sequence $r < R_1 < R_2 < R_3 < \cdots \to \infty$ (set $a_k := a(R_k)$) such that $f(z+a_k)$ converges to some $g(z)$ in $\mathcal{M}(X)$. (Then $g \in \overline{\mathbf{C}\cdot f}$.) We have

$$\inf_{r\le t\le R_k}\frac{1}{\pi t^2}\int_{B_t}|df|^2(z+a_k)dxdy > \rho(f) - \varepsilon.$$

Hence for any $t \ge r$ we get

$$\frac{1}{\pi t^2}\int_{B_t}|dg|^2(z)dxdy \ge \rho(f) - \varepsilon.$$

Then for $s \ge r$ (> 1)

$$\begin{aligned}T(s,g) &\ge \int_r^s\left(\int_{B_t}|dg|^2 dxdy\right)\frac{dt}{t} \\ &\ge (\rho(f)-\varepsilon)\left(\frac{\pi s^2}{2} - \frac{\pi r^2}{2}\right).\end{aligned}$$

Hence for $s \ge r$

$$\frac{2}{\pi s^2}T(s,g) \ge (\rho(f)-\varepsilon)\left(1 - \frac{r^2}{s^2}\right).$$

Taking the limit-inf with respect to s, we get $\underline{\rho}_{\mathrm{NSA}}(g) \ge \rho(f) - \varepsilon$. Thus

$$\underline{\rho}_{\mathrm{NSA}}(\overline{\mathbf{C}\cdot f}) \ge \underline{\rho}_{\mathrm{NSA}}(g) \ge \rho(f) - \varepsilon.$$

$\varepsilon > 0$ is arbitrary. So $\underline{\rho}_{\mathrm{NSA}}(\overline{\mathbf{C}\cdot f}) \ge \rho(f)$. □

Remark 3.1. By using the above argument, we can also prove that $\rho(f)$ is equal to the supremum of

$$\limsup_{r\to+\infty}\left(\frac{1}{\pi r^2}\int_{|z|<r}|dg|^2 dxdy\right)$$

over $g\in\overline{\mathbf{C}\cdot f}$. (The limit-sup can be replaced with the limit-inf.) This type of energy density was introduced and studied in [7].

Proof of Theorem 1.3. Let ρ be the right-hand-side of (1). $\rho\geq\rho(\mathcal{N})$ is obvious (by the $\mathbf{C}$-invariance of $\mathcal{N}$). For each $f\in\mathcal{N}$ we define a Borel measure μ_f on $\mathbf{C}$ by $\mu_f(\Omega):=\int_\Omega|df|^2dxdy$. Consider the set $\{\mu_f\mid f\in\mathcal{N}\}$. This set satisfies the conditions (a) and (b) in the beginning of Section 2. Then Theorem 2.2 implies that ρ is equal to

$$\lim_{r\to\infty}\left[\lim_{R\to\infty}\left\{\sup_{f\in\mathcal{N}}\left(\inf_{r\leq t\leq R}\frac{\int_{B_t}|df|^2dxdy}{\pi t^2}\right)\right\}\right].$$

Then, as in the proof of Step 2, for every $\varepsilon>0$ we can find $r_\varepsilon>0$ and $g_\varepsilon\in\mathcal{N}$ such that for all $t\geq r_\varepsilon$

$$\frac{1}{\pi t^2}\int_{B_t}|dg_\varepsilon|^2dxdy\geq\rho-\varepsilon.$$

Then $\rho(\mathcal{N})\geq\rho(g_\varepsilon)\geq\rho-\varepsilon$. Since $\varepsilon>0$ is arbitrary, we get $\rho(\mathcal{N})\geq\rho$. $\square$

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