p-adic properties of coefficients of basis for the space of weakly holomorphic modular forms of weight 2

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Abstract: We observe properties of coefficients of certain basis elements for the space of weakly holomorphic modular forms of weight 2 for $SL_2(\mathbf{Z})$. Moreover we show that these coefficients are often highly divisible by the primes 2, 3, 5, 7, 11.

Key words: Weakly holomorphic modular form; congruence.

1. Introduction. Let k be any even integer. A weakly holomorphic modular form of weight k for $SL_2(\mathbf{Z})$ is a holomorphic function on the upper half plane $\mathbf{H}$, but may have poles at the cusp ∞ which satisfies the modular transformation

$$f(\gamma z) = (cz+d)^k f(z) \text{ for any } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}).$$

Since $SL_2(\mathbf{Z})$ has only one cusp, for each even integer k there is a canonical basis for the space $M_k^!$ of weakly holomorphic modular forms of weight k, indexed by the order of the pole at ∞. To be more precise, write $k = 12l + k'$ with $k' \in \{0, 4, 6, 8, 10, 14\}$. Then for each integer $m \geq -l$, Duke and Jenkins [3] showed that there exists a unique weakly holomorphic modular form $f_{k,m}$ of weight k with a q-expansion of the form

$$f_{k,m}(z) = q^{-m} + O(q^{l+1}).$$

Throughout this paper $q = e^{2\pi i z}$. Since for all non-zero $f \in M_k^!$ we have $\mathrm{ord}_\infty(f) \leq l$, the functions $f_{k,m}$ form a basis for $M_k^!$. Indeed, they constructed the basis elements $f_{k,m}$ from the classical discriminant function Δ, the modular invariant j and the Eisenstein series $E_{k'}$ (we let $E_0 = 1$) as follows: We recall

$$\Delta(z) = q\prod_{n\geq 1}(1-q^{24})^n = \sum_{n\geq 1}\tau(n)q^n,$$

$$E_r(z) = 1 - \frac{2r}{B_r}\sum_{n=1}^{\infty}\sigma_{r-1}(n)q^n$$

and

$$j(z) = E_4(z)^3/\Delta(z) = \sum_{n\geq -1} c(n)q^n,$$

where B_r is the r-th Bernoulli number and σ_{r-1} stands for the usual divisor sum. We have that $f_{k,-l} = \Delta(z)^l E_{k'}$. Now for each $n \geq 1$, we obtain $f_{k,-l+n}$ by multiplying $f_{k,-l+n-1}$ by j and then substracting off multiples of $f_{k,-l+d}$ in order to kill successively the coefficients of q^{l-d} for $0 \leq d \leq n-1$. This construction shows that

$$f_{k,m} = \Delta^l E_{k'} F_{k,D}(j),$$

where $F_{k,D}(x)$ is a monic polynomial in x of degree $D = j + m$ with integer coefficients. Motivated by work of Zagier, the forms $f_{k,0}$ play an important role in the study of singular moduli and the polynomials $F_{k,D}(x)$ are a generalization of the classical Faber polynomials $F_{0,m}(x)$.

Throughout this paper we define the Fourier coefficients $a_k(m,n)$ of these basis elements $f_{k,m}$ by

$$f_{k,m}(z) = q^{-m} + \sum_{n>l} a_k(m,n)q^n.$$

Here we note that the coefficients $a_k(m,n)$ are integral.

Noticing $f_{12,-1} = \Delta$ and $f_{0,1} = j - 744$ we know that Ramanujan [8] showed $a_{12}(-1,2n) \equiv 0 \pmod 2$, $a_{12}(-1,3n) \equiv 0 \pmod 3$, $a_{12}(-1,5n) \equiv 0 \pmod 5$ and Lehner [6,7] showed

$$a_0(1, 2^a 3^b 5^c 7^d 11n) \equiv 0 \pmod{2^{3a+8}3^{2b+3}5^{c+1}7^d 11}.$$

Recently Duke and Jenkins [3] studied congruence properties of the basis elements $f_{k,m}$. In particular they showed the following

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Theorem 1.1 [3, Corollary 1]. *For any even integer k and any integers m, n we have that*

$$a_k(m,n) = -a_{2-k}(n,m).$$

Theorem 1.2 [3, Lemma 1]. *Let p ba a prime and* $k \in \{4,6,8,10,14\}$*. Then for* $m,n,s \in \mathbf{Z}$*, with* $n,m,s>0$ *we have that*

$$a_k(m,np^s) = p^{s(k-1)}(a_k(mp^s,n) - a_k(mp^{s-1},n/p)) + a_k(m/p,np^{s-1}).$$

By using Theorem 1.1 and Theorem 1.2, Doud and Jenkins [2, Theorem 1.3] proved that the coefficients $a_k(m,n)$ are often highly divisible by the primes 2, 3, 5 when $k \in \{4,6,8,10,14\}$. In this paper we observe divisibility properties of the coefficients $a_2(m,n)$.

For each prime p, the Hecke operator T_p for weight 2 weakly holomorphic modular forms to weight 2 weakly holomorphic modular forms is defined by

$$(f_{2,m}|T_p)(z) = \sum_n \left(a_2(m,np) + pa_2\left(m,\frac{n}{p}\right)\right)q^n,$$

where $a_2(m,\frac{n}{p}) = 0$ if p does not divide n. Since there is no holomorphic modular form of weight 2 for $SL_2(\mathbf{Z})$ and the functions $f_{k,m}$ form a basis for $M_k^!$, following the argument in [3] we obtain

$$a_2(m,np) = p\left(a_2(mp,n) - a_2\left(m,\frac{n}{p}\right)\right) + a_2\left(\frac{m}{p},n\right). \tag{1}$$

By (1) and the same arguments in [3] we obtain the following proposition.

Proposition 1.3. *For each prime p and any positive integers n, m, s we have that*

$$a_2(m,np^s) = p^s\left(a_2(mp^s,n) - a_2\left(mp^{s-1},\frac{n}{p}\right)\right) + a_2\left(\frac{m}{p},np^{s-1}\right).$$

Applying induction to this proposition, we obtain the following

Corollary 1.4. *Let* $(m,p)=(n,p)=1$*,* $r>0$ *and* $s \geq 1$*. Then for* $0 \leq t \leq min(r,s-1)$*, we have that*

$$a_2(mp^r,np^s) = a_2(mp^{r-t-1},np^{s-t-1}) + \sum_{j=0}^{t} p^{(s-j)}a_2(mp^{r+s-2j},n).$$

Proposition 1.3 also implies the following corollary.

Corollary 1.5. *If* $p^r|n$ *and* $p \nmid m$ *then* $p^r|a_2(m,n)$*. In particular, if* $(m,n)=1$*, we have* $n|a_2(m,n)$*.*

In this paper by combining ideas of Doud and Jenkins [2] with ideas of Lehner [6,7] we prove the following theorems making above divisibility results more explicit. For each integer N, let $v_p(N)$ be the largest integer s such that $p^s|N$.

Theorem 1.6. *We have the following inequalities: For all positive integers m, n,*

(i)

$$v_2(a_2(m,n)) \geq \begin{cases} 3(v_2(m)-v_2(n))+8 & \text{if } v_2(m)>v_2(n) \\ 4(v_2(n)-v_2(m))+8 & \text{if } v_2(n)>v_2(m). \end{cases}$$

(ii)

$$v_3(a_2(m,n)) \geq \begin{cases} 2(v_3(m)-v_3(n))+3 & \text{if } v_3(m)>v_3(n) \\ 3(v_3(n)-v_3(m))+3 & \text{if } v_3(n)>v_3(m). \end{cases}$$

(iii)

$$v_5(a_2(m,n)) \geq \begin{cases} v_5(m)-v_5(n)+1 & \text{if } v_5(m)>v_5(n) \\ 2(v_5(n)-v_5(m))+1 & \text{if } v_5(n)>v_5(m). \end{cases}$$

(iv)

$$v_7(a_2(m,n)) \geq \begin{cases} v_7(m)-v_7(n) & \text{if } v_7(m)>v_7(n) \\ 2(v_7(n)-v_7(m)) & \text{if } v_7(n)>v_7(m). \end{cases}$$

(v)

$$v_{11}(a_2(m,n)) \geq \begin{cases} 1 & \text{if } v_{11}(m)>v_{11}(n) \\ v_{11}(n)-v_{11}(m)+1 & \text{if } v_{11}(n)>v_{11}(m). \end{cases}$$

Remark 1.7. By the duality $a_0(n,m) = -a_2(m,n)$ (Theorem 1.1), The Theorem 1.6 also gives the corresponding results for $a_0(n,m)$.

2. Preliminaries. Let p be a prime, and $\Gamma_0(p)$ be the subgroup of $SL_2(\mathbf{Z})$ consisting of elements γ with $\gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod p$. For a weakly holomorphic modular form f of weight k for $SL_2(\mathbf{Z})$ we introduce the linear operator

$$U_p f(z) = \frac{1}{p}\sum_{\lambda=0}^{p-1} f\left(\frac{z+\lambda}{p}\right).$$

It is well known [1, Theorem 4.5] [4, Propersition 2.22] that U_pf is a weakly holomorphic modular form of weight k for $\Gamma_0(p)$ and if $f(z) = \sum_{n\geq s} a_n q^n$ then

$$f_p := U_p f = \sum_{n\geq[s/p]} a_{pn} q^n.$$

We denote $U_p(U_p^a f)$ by $U_p^{a+1} f$ for each positive integer a, where $U_p^1 f = U_p f$.

Lemma 2.1. [2, Corollay 4.2] *Let f ba a weakly holomorphic modular form of weight k for $SL_2(\mathbf{Z})$. Then*

$$p(pz)^{-k} f_p(-1/(pz)) = -f(z) + pf_p(pz) + p^k f(p^2 z).$$

Further, $p(pz)^{-k} f_p(-1/(pz))$ is a weakly holomorphic modular form of weight k on $\Gamma_0(p)$.

Since the subgroups $\Gamma_0(p)$ are of genus zero for the primes $p \in \{2, 3, 5, 7\}$, they have univalent functions, which may [6,7] be taken as

$$\Phi(z) = \Phi_{p,r}(z) = \left(\frac{\eta(pz)}{\eta(z)}\right)^r = q + \cdots,$$

with

$$\eta(z) = q^{\frac{1}{24}} \prod_{n\geq1}(1 - q^n),$$

and

$$r(p-1) = 24.$$

Let $j_p(z) = 1/\Phi_{p,r}(z)$. Then we have that j_p is holomorphic on the upper half plane $\mathbf{H}$, has a simple pole at the cusp ∞ and

$$(2) \qquad j_p(-1/(pz)) = p^{r/2}\Phi_{p,r}(z).$$

For (2) see [5, (8.83)]. Indeed by the transformation law of η we can easily show (2). Moreover j_p and Φ have integral Fourier coefficients.

From now on, for each positive integer m we let

$$f(z) = f_{0,m}(z) = \frac{1}{q^m} + O(q)$$

and assume that the prime p does not divide m. Then f_p is holomorphic on $\mathbf{H}$ and at the cusp ∞. Moreover from Lemma 2.1 we have

$$pf_p(-1/(pz)) = -f(z) + pf_p(pz) + f(p^2 z),$$

which is a weakly holomorphic modular form of weight 0 for $\Gamma_0(p)$, holomorphic at the cusp 0 and meromorphic at the cusp ∞ having integral Fourier coefficients in the q-expansion at ∞. Thus for each prime $p \in \{2, 3, 5, 7\}$, there exist integers $A_{t,p}$ such that

$$pf_p(-1/(pz)) = \sum_{t\geq0} A_{t,p} j_p(z)^t.$$

Replacing z by $-1/(pz)$, we obtain the following theorem.

Theorem 2.2. *For each prime $p \in \{2, 3, 5, 7\}$, there exist integers $D_t = D_{t,p}$ such that*

$$f_p(z) = D_{0,p} + \sum_{t\geq1} D_{t,p} p^{rt/2-1} \Phi(z)^t.$$

3. Proofs of Theorems. In this section we use the same notations and assumptions in Section 2. We first prove Theorem 1.6(i).

Proof. Let $p = 2$. Then $r = 24$ and we can rewrite f_2 in Theorem 2.2 as

$$(3) \quad f_2 = B_0 + 2^{11} \sum_{t\geq1} B_t 2^{8(t-1)} \Phi^t = B_0 + 2^{11} R,$$

where R is a polynomial of the form

$$R = \sum_{t\geq1} d_t 2^{8(t-1)} \Phi^t$$

with integers d_t. R will denote a polynomial of this type, not necessarily the same one at each appearance. Applying the operator U_2 to both sides in (3) we obtain

$$(4)\ U_2^2 f = B_0 + 2^{11} \sum_{t\geq1} B_t 2^{8(t-1)} U_2 \Phi^t = B_0 + 2^{11} U_2 R.$$

In the above equations $B_t's$ are integers.

Proposition 3.1. *For each positive integer h, we have that $2^{8(h-1)} U_2 \Phi^h = 2^3 R$.*

Proof. See [7, (3.4)] □

This proposition implies that for each positive integer a,

$$U_2^a f = A_0 + 2^{11} 2^{3(a-1)} R \equiv A_0 \pmod{2^{3a+8}},$$

which says

$$(5) \quad a_2(2^a n, m) \equiv -a_0(m, 2^a n) \equiv 0 \pmod{2^{3a+8}}.$$

Now in Corollary 1.4 if $r > s$ then take $t = s - 1$. Thus for $(m, 2) = (n, 2) = 1$, $r > 0$ and $s \geq 1$, from (5) we have that

$$a_2(m2^r, n2^s) = a_2(m2^{r-s}, n) + \sum_{j=0}^{s-1} 2^{(s-j)} a_2(m2^{r+s-2j}, n) \equiv 0 \pmod{2^{3(r-s)+8}}.$$

If $r < s$ then take $t = r$ in Corollary 1.4. Thus for $(m,2) = (n,2) = 1$, $r > 0$ and $s \geq 1$, from (5) we have that

$$a_2(m2^r, n2^s) = \sum_{j=0}^{r} 2^{(s-j)} a_2(m2^{r+s-2j}, n) \equiv 0 \pmod{2^{4(s-r)+8}}$$

which implies the assertion. □

We prove Theorem 1.6(ii).

Proof. Let $p = 3$. Then $r = 12$ and we can rewrite f_3 in Theorem 2.2 as

$$f_3 = B_0 + 3^5 \sum_{t\geq 1} B_t 3^{4(t-1)} \Phi^t. \tag{6}$$

Proposition 3.2. *For each positive integer h, we have that $3^{4(h-1)} U_3 \Phi^h = 3^2 T$, where T is a polynomial of the form $T = \sum_{t\geq 1} d_t 3^{4(t-1)} \Phi^t$ with integers d_t.*

Proof. See [7, (3.24)] □

This proposition implies that for each positive integer a,

$$U_3^a f = A_0 + 3^{2a+3} T \equiv A_0 \pmod{3^{2a+3}},$$

which says

$$a_2(3^a n, m) \equiv -a_0(m, 3^a n) \equiv 0 \pmod{3^{2a+3}}. \tag{7}$$

Now in Corollary 1.4 if $r > s$ then take $t = s - 1$. Thus for $(m,3) = (n,3) = 1$, $r > 0$ and $s \geq 1$, from (7) we have that

$$a_2(m3^r, n3^s) = a_2(m3^{r-s}, n) + \sum_{j=0}^{s-1} 3^{(s-j)} a_2(m3^{r+s-2j}, n) \equiv 0 \pmod{3^{2(r-s)+3}}.$$

If $r < s$ then take $t = r$. Thus for $(m,3) = (n,3) = 1$, $r > 0$ and $s \geq 1$, from (5) we have that

$$a_2(m3^r, n3^s) = \sum_{j=0}^{r} 3^{(s-j)} a_2(m3^{r+s-2j}, n) \equiv 0 \pmod{3^{3(s-r)+3}}$$

which implies the assertion. □

We prove Theorem 1.6(iii).

Proof. Let $p = 5$. Then $r = 6$ and we can rewrite f_5 in Theorem 2.2 as

$$f_5 = B_0 + \sum_{t\geq 1} B_t 5^{3t-1} \Phi^t = B_0 + 5^2 Q, \tag{8}$$

where Q is a polynomial of the form $Q = b_1\Phi + \sum_{t\geq 2} b_t 5^t \Phi^t$ with integers b_l.

Proposition 3.3. *For each positive integer $h > 1$, we have that $U_5\Phi = 5Q$ and $5^h U_5 \Phi^h = 5Q$, where Q is a polynomial of the form $Q = b_1\Phi + \sum_{t\geq 2} b_t 5^t \Phi^t$ with integers b_l.*

Proof. See [6, (5.13), (5.14)] □

This proposition implies that for each positive integer a,

$$U_5^a f = A_0 + 5^{a+1} Q \equiv A_0 \pmod{5^{a+1}},$$

which says

$$a_2(5^a n, m) \equiv -a_0(m, 5^a n) \equiv 0 \pmod{5^{a+1}}. \tag{9}$$

Now similar method in the proof of Theorem 1.6(i) show the assertion. □

We prove Theorem 1.6(iv).

Proof. Let $p = 7$. Then $r = 4$ and we can rewrite f_7 in Theorem 2.2 as

$$f_7 = B_0 + \sum_{t\geq 1} B_t 7^{2t-1} \Phi^t = B_0 + Q, \tag{10}$$

where Q is a polynomial of the form $Q = b_1\Phi + \sum_{t\geq 2} b_t 7^t \Phi^t$ with integers b_l.

Proposition 3.4. *For each positive integer $h > 1$, we have that $U_7\Phi = 7Q$ and $7^h U_7 \Phi^h = 7Q$, where Q is a polynomial of the form $Q = b_1\Phi + \sum_{t\geq 2} b_t 7^t \Phi^t$ with integers b_l.*

Proof. See [6, Section 6] □

This proposition implies that for each positive integer a,

$$U_7^a f = A_0 + 7^a Q \equiv A_0 \pmod{7^a},$$

which says

$$a_7(7^a n, m) \equiv a_0(m, 7^a n) \equiv 0 \pmod{7^a}. \tag{11}$$

Now similar method in the proof of Theorem 1.6(i) show the assertion. □

Lastly we prove Theorem 1.6(v). Since the genus of $\Gamma_0(11)$ is not zero, we face a new situation. We need another modular form instead of j_p as follows: Following the notation in [5] we have modular functions for $\Gamma_0(11)$ which are holomorphic on $\mathbf{H}$ and have integral Fourier coefficients [5, (4.51), (6.44), (6.46) and Lemma 3] as follows:

$$A(z) = A\left(\frac{-1}{11z}\right) = \frac{1}{q} + 6 + 17q + 46q^2 + \cdots,$$

$$C(z) = q + 5q^2 + \cdots,$$

$$11^2 C\left(\frac{-1}{11z}\right) = \frac{1}{q^2} + \frac{2}{q} + \cdots.$$

Letting

$$\alpha(z) = 11^2 C\left(\frac{-1}{11z}\right) = \frac{1}{q} + \cdots,$$
$$\beta(z) = 11^2 C\left(\frac{-1}{11z}\right) A(z) = \frac{1}{q^2} + \cdots,$$

we come up with

$$11 f_{11}\left(\frac{-1}{11z}\right) = \sum_{a\geq 0, b\geq 0} D_{ab}\alpha(z)^a \beta(z)^b$$

for some integers D_{ab}. Now replacing z by $-1/11z$ we obtain that

$$11 f_{11}(z) = \sum_{a\geq 0, b\geq 0} D_{ab}\alpha\left(\frac{-1}{11z}\right)^a \beta\left(\frac{-1}{11z}\right)^b$$
$$= \sum_{a\geq 0, b\geq 0} D_{ab} 11^{2(a+b)} C(z)^{a+b} A(z)^b,$$

which implies that $f_{11}(z) \equiv A_0 \pmod{11}$ for some integer A_0 and hence $a_2(11n, m) = -a_0(m, 11n) \equiv A_0 \pmod{11}$.

Now in Corollary 1.4 if $r > s$ then take $t = s - 1$ and if $r < s$ then take $t = r$. Then by the same argument in the above case we have the assertion.

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