On Kaufhold's Whittaker functions

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Abstract: In this papr we give an integral representation for a Whittaker function of an non holomorphic Eisenstein series which is a non holomorphic Sigel modular form of degree 2. Our integral representation is very useful to the theory of the theta lifting of automorphic forms.

Key words: Siegel modular; theta correspondence; generalized Whittaker functions.

1. We are concerned with the Whittaker functions of non holomorphic Eisenstein series which are non holomorphic Siegel modular forms of degree 2. Such functions, which we call Kaufhold's Whittaker functions in this paper, were first investigated by Kaufhold [4] and were generalized to higher degree cases by Shimura [6]. The purpose of this paper is to give an integral representation for Kaufhold's Whittaker fuction with complete and elementary proof. We note that our representation is very useful in the theory of the theta lifting of automorphic forms. Let $M_n(\mathbf{R})$ denote the set of all real square matrices of degree n. Put $W_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and

(1) $$\mathcal{P} = \{V \in M_2(\mathbf{R}) \mid {}^tV = V, V > 0\}.$$

Then one of Kaufhold's Whittaker function is the function

(2) $$W(Y) = |Y|^s \exp(-2\pi \operatorname{tr} Y) h_{20}(4\pi Y, s/2, s/2)$$

of $Y \in \mathcal{P}$ where

(3) $$h_{20}(L, \alpha, \beta) = \int_{\mathcal{P}} |V + E|^{\alpha - 3/2} |V|^{\beta - 3/2} \exp(-\operatorname{tr} VL) dV,$$

which is $4^{3/2-s} \exp(2^{-1} \operatorname{tr} L)$ times the right hand side of Kaufhold [4, (1,14)] if $\alpha = \beta = s/2$, $n = h = p = 2$, $H = Q = E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Theorem 1. *The function $W(Y)$ of $Y \in \mathcal{P}$ has the following integral representation:*

(4) $$W(Y) = 2^{s-3}\Gamma(s-1)^{-1}|Y| \int_{-\infty}^{\infty} \int_{\mathcal{P}} \exp(-\pi \operatorname{tr}(((1+u^2)V + V^{-1} + u|V|^{-1/2}(W_0V - VW_0))Y)) |V|^{s/2-2} dV du.$$

Denote $2^{-s+3}\Gamma(s-1)$ times the right hand side of (4) by $F(Y)$. $F(Y)$ is closely connected to the Bessel function $K_\delta(Z)$ in Herz [3, p. 517]. We shall give the proof of Theorem 1 in the next sections.

2. Put $g[x] = {}^txgx$ as usual. We denote the Siegel upper half space of degree n by $\mathbf{H}_n$, i.e.,

(5) $$\mathbf{H}_n = \{Z = U + iV \in M_n(\mathbf{C}) \mid U, V \in M_n(\mathbf{R}), {}^tZ = Z, V > 0\}.$$

Since we can easily see that $F(Y[k]) = F(Y)$, $W(Y[k]) = W(Y)$ for every $Y \in \mathcal{P}$, $k \in SO(2)$, the functions F, W are determined by the values $F(Y), W(Y)$ for $Y = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$. Since the function $\exp(2\pi i \operatorname{Re} Z) W(\operatorname{Im} Z)$ of $Z \in \mathbf{H}_2$ is a Fourier coefficient of an Eisenstein series, it is an eigenfunction for the invariant differential operators Δ_1, Δ_2 in Niwa [5] with eigenvalues

(6) $$d_1 = s(s-3)/8, \quad d_2 = s(s+1)(s-3)(s-4)/256.$$

To prove Theorem 1, it suffices to show that $\exp(2\pi i \operatorname{Re} Z) F(\operatorname{Im} Z)$ is also an eigenfunction for Δ_1, Δ_2 with eigenvalues d_1, d_2 and that the first term (the constant term) of the Maclaurin expansion of $W\left(\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}\right)$ in $t_1 - t_2$ is equal to that of $F\left(\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}\right)$. For $-y < x < y$ put

(7) $$F\left(\begin{pmatrix} (y+x)/2 & 0 \\ 0 & (y-x)/2 \end{pmatrix}\right) = (x^2 - y^2) \sum_{k=0}^{\infty} a_k x^k,$$

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(8) $$W\left(\begin{pmatrix} (y+x)/2 & 0 \\ 0 & (y-x)/2 \end{pmatrix}\right) = (x^2-y^2)\sum_{k=0}^{\infty} b_k x^k$$

where a_k, b_k are functions of y. Then it follows from

(9) $$\Delta_1 \exp(2\pi i \operatorname{Re} Z) W(\operatorname{Im} Z) = d_1 \exp(2\pi i \operatorname{Re} Z) W(\operatorname{Im} Z)$$

that

(10) $$\frac{1}{8}(-2b_k + 2kb_k + k^2 b_k - 4\pi^2 y^2 b_k - 4\pi^2 b_{k-2} + 4y^2 b_{k+2} + 4ky^2 b_{k+2} + k^2 y^2 b_{k+2} + 4yb'_k + 4kyb'_k + y^2 b''_k + b''_{k-2}) = d_1 b_k$$

which is exactly the same as Niwa [5, (1.9.1)]. It is easy to see $a_k = 0, b_k = 0$ for odd k. The following two propositions accomplish the proof of Theorem 1.

Proposition 1.

(11) $$\frac{1}{8}(-2a_k + 2ka_k + k^2 a_k - 4\pi^2 y^2 a_k - 4\pi^2 a_{k-2} + 4y^2 a_{k+2} + 4ky^2 a_{k+2} + k^2 y^2 a_{k+2} + 4ya'_k + 4kya'_k + y^2 a''_k + a''_{k-2}) = d_1 a_k.$$

Proposition 2.

(12) $$b_0 = 2^{-4-s}\pi^{-1}\Gamma(s-1)a_0.$$

3. We shall prove Proposition 1 in this section. We assume $\operatorname{Re} r > 2\operatorname{Re} s > 10 + 4k$ throughout this paper. All multiple integrals converge absolutely and we can change the order of the integrations. Define the Mellin transform of $f(y)$ by

(13) $$M(f, r) = \int_0^{\infty} f(y) y^{r-1} dy.$$

Then, to prove the equalities (11), (12), it suffices to show that the Mellin transforms of the both sides of them are the same respectively. By definition we have

(14) $$M(a_{2k}, r) = \int_0^{\infty}\int_{-\infty}^{\infty}\int_{\mathcal{P}} \exp(-\pi \operatorname{tr}(((1+u^2)V + V^{-1})y/2E))\pi^{2k} \left(\operatorname{tr}\left(1/2((1+u^2)V + V^{-1})\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) + 2|V|^{-1/2}(1 \quad 0)V\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)^{2k} |V|^{s/2-2} dV du y^{r-1} dy,$$

and therefore, by changing variables

(15) $$V \to (1+u^2)^{-1/2} V$$

and

(16) $$V \to \begin{pmatrix} e^{\theta}\cosh\psi & e^{(\theta+\varphi)/2}\sinh\psi \\ e^{(\theta+\varphi)/2}\sinh\psi & e^{\varphi}\cosh\psi \end{pmatrix},$$

we have

(17) $$a_{2k} = -\frac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (1+u^2)^{(1-s)/2}\exp((\theta+\varphi)(s-1)/2) \cosh\psi \exp(-\pi(1+u^2)^{1/2} y (\cosh\theta + \cosh\varphi)\cosh\psi) ((\cosh\theta - \cosh\varphi)\cosh\psi + 2u(1+u^2)^{-1/2}\sinh\psi)^{2k} \pi^{2k}/(2k)!\, du\, d\theta\, d\varphi\, d\psi = \sum_{l=0}^{k}\binom{2k}{2l}\pi^{2k} a_{2k,2l}/((2k)!)$$

where

(18) $$a_{2k,2l} = -\frac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (1+u^2)^{(1-s+2k-2l)/2}\exp((\theta+\varphi)(s-1)/2) \cosh\psi\exp(-\pi(1+u^2)^{1/2} y (\cosh\theta+\cosh\varphi)\cosh\psi) ((\cosh\theta-\cosh\varphi)\cosh\psi)^{2k-2l} (2u\sinh\psi)^{2l} du\, d\theta\, d\varphi\, d\psi.$$

By using the binomial expansion of $((\cosh\theta - \cosh\varphi)\cosh\psi)^{2k-2l}$, the integral representation $\frac{1}{2}\int_0^{\infty}\exp(-\nu t - z\cosh t)dt$ for the modified Bessel function $K_{\nu}(z)$ and changing variables $y \to y/(\pi(1+u^2)^{1/2}\cosh\psi)$, we have

(19) $$M(a_{2k,2l}, r) = -\frac{1}{4}\sum_{j=0}^{2k-2l} M_{k,l,j}\binom{2k-2l}{j}$$

where

(20) $$M_{k,l,j} = 4(-1)^j \pi^{-r} \int_0^{\infty} K_{(s-1)/2}^{(2k-2l-j)}(y) K_{(s-1)/2}^{(j)}(y) y^{r-1} dy \int_{-\infty}^{\infty}(1+u^2)^{(1-s-r+2k-2l)/2}(2u)^{2l} du \int_{-\infty}^{\infty}(\cosh\psi)^{1-r+2k-2l}(\sinh\psi)^{2l} d\psi.$$

(21) $$M(a_{2k,2l},r)=2^{2l-1}\Gamma\left(l+\frac{1}{2}\right)^2 \Gamma\left(\frac{1}{2}(-2k+r-1)\right)\Gamma\left(-k+l+\frac{r}{2}\right)^{-1} \Gamma\left(\frac{1}{2}(-2k+2l+r+s-1)\right)^{-1} \Gamma\left(\frac{1}{2}(-2k+r+s-2)\right)I(2k-2l,r,s)$$

where

(22) $$I(n,r,s)=\sum_{j=0}^{n}\binom{n}{j}(-1)^j \int_0^\infty K^{(n-j)}_{(s-1)/2}(y)K^{(j)}_{(s-1)/2}(y)y^{r-1}dy.$$

Proposition 3. *For even positive n,*

(23) $$I(n,r,s)=\Gamma\left(\frac{r}{2}-\frac{n}{2}\right)n!\sqrt{\pi} \sum_{i=0}^{n}(-1)^i\binom{n}{i}\Gamma\left(\frac{1}{2}(-2i+r+n-s+1)\right) \Gamma\left(\frac{1}{2}(2i+r-n+s-1)\right) \left(\frac{n}{2}!2^{n+2}\Gamma\left(\frac{r+1}{2}\right)\right)^{-1}.$$

Proof. Put

(24) $$H(n,r,s,t)=\sum_{j=0}^{n}\binom{n}{j}(-1)^j \int_0^\infty K^{(n-j)}_{(s-1)/2}(y)K^{(j)}_{(t-1)/2}(y)y^{r-1}dy$$

and

(25) $$K(n,r,s,t)=2^{-n-3+r}/\Gamma(r) \left(\sum_{k=0}^{n}\Gamma((2n-4k+2r+s-t)/4) \Gamma((-2n+4k+2r-s+t)/4)(-1)^k\binom{n}{k}\right) \left(\sum_{l=0}^{n}\Gamma((2n-4l+2r-s-t+2)/4) \Gamma((-2n+4l+2r+s+t-2)/4)(-1)^l\binom{n}{k}\right).$$

Then this proposition follows from the next proposition by putting $t=s$. □

Proposition 4. *For every non negative integer n,*

(26) $$H(2n,r,s,t)=K(2n,r,s,t).$$

Proof. For $n=0$ see Gradshteyn and Ryzhik [2, 6.576-4]. We can easily prove this proposition by induction on n using the following recurrence relation.

(27) $$\begin{aligned}&4H(2n+2,r,s,t)\\&=H(2n,r,s+4,t)+2H(2n,r,s,t)\\&\quad+H(2n,r,s-4,t)+H(2n,r,s,t+4)\\&\quad+2H(2n,r,s,t)+H(2n,r,s,t-4)\\&\quad-2(H(2n,r,s+2,t+2)+H(2n,r,s+2,t-2)\\&\quad+H(2n,r,s-2,t+2)+H(2n,r,s-2,t-2))\end{aligned}$$

which is proved by $\binom{n+2}{k}=\binom{n}{k}+2\binom{n}{k-1}+\binom{n}{k-2}$. (The same relation holds for K). □

By using $\Gamma(s+1)=s\Gamma(s)$ succesively, we easily get the following

Proposition 5. *For every non negative integer n,*

(28) $$I(2n,r,s)=\sqrt{\pi}(2n)!2^{-2n-2}r_n(r,s) \Gamma(r/2-n)\Gamma((r-2n+s-1)/2) \Gamma((r-2n-s+1)/2)/(n!\Gamma((r+1)/2))$$

where $r_n(r,s)$ is the following polynomial in r,s of degree $2n$:

(29) $$r_n(r,s)=\sum_{i=0}^{2n}(-1)^i\binom{2n}{i} \left(\prod_{j=0}^{-i+2n-1}\left(j-n+\frac{r-s}{2}+\frac{1}{2}\right)\right) \prod_{j=1}^{i}\left((j-1)-n+\frac{r+s}{2}-\frac{1}{2}\right).$$

By definition we immediately get

Proposition 6. *For every non negative integer n,*

(30) $$r_n(0,2-s)=r_n(0,s).$$

We easily get also

Proposition 7. *For every integer n,x such that $0\le x\le n$,*

(31) $$r_n(0,-2x+1)=(2n)!(-1)^{n+x}.$$

It is easy to see that the definition of $r_n(r,s)$ deduces

Proposition 8. *For every non negative integer n,*

(32) $$4r_{n+1}(r,s)=r_n(r,s-2)$$

$$(-2n-1+r-s)(-2n+1+r-s) + r_n(r,s+2)(-2n-1+r+s) (-2n-3+r+s) - 2r_n(r,s) (-2n-1+r-s)(-2n-3+r+s).$$

The induction on n using (32) easily proves the following

Proposition 9. *For every non negative integer n,*

$$(33)\quad (-2n+1+r-s)r_n(r,s-2) + (-2n-1+r+s)r_n(r,s+2) - 2(2n+r)r_n(r,s) = 0.$$

Proposition 10. *For every positive integer n,*

$$(34)\quad r_n(r+2,s) - r_n(r,s) = 2n(2n-1)r_{n-1}(r,s).$$

Proof. We prove this proposition by induction on n. Using (32) and the induction assumption

$$r_n(r+2,s) - r_n(r,s) = 2n(2n-1)r_{n-1}(r,s),$$

we have

$$(35)\quad r_{n+1}(r+2,s) - r_{n+1}(r,s) - (2n+2)(2n+1)r_n(r,s) = (-2n-2+r+2-s)(-2n-2+r+2-s) r_n(r+2,s-2)/4 + (-2n-3+r+2+s)(-2n-2+r+2+s) r_n(r+2,s+2)/4 - (-2n-2+r+2-s)(-2n-3+r+2+s) r_n(r+2,s)/2 - ((-2n-2+r-s)(-2n-2+r-s) r_n(r,s-2)/4 + (-2n-3+r+s)(-2n-2+r+s) r_n(r,s+2)/4 - (-2n-2+r-s)(-2n-3+r+s)r_n(r,s)/2) - (2n+2)(2n+1)r_n(r,s) = (-2n+1+r-s)r_n(r,s-2) + (-2n-1+r+s)r_n(r,s+2) - 2(2n+r)r_n(r,s)$$

which is 0 by (33). □

(34) proves that $r_n(r,s)$ is a polymonial in r of degree n.

The following equality follows from (17), (21), (23).

$$(36)\quad M(a_{2k},r) = -2^3\pi^{3/2+2k-r}2^{-2k}(k!)^{-1} \Gamma((-2-2k+r+s)/2)\Gamma(-k+(r-1)/2) \Gamma((1-2k+r-s)/2)R_k(r,s)/\Gamma((1+r)/2)$$

where

$$(37)\quad R_k(r,s) = \frac{1}{(2k)!}\left(2^{2k}k!\right) \sum_{l=0}^{k} \frac{1}{l!2^{2l}} \left(\prod_{j=0}^{k-l-1}(1+2j)\right)^2 \left(\prod_{j=0}^{k-l-1}\left(-k+\frac{1}{2}+\frac{r-s}{2}+j\right)\right) \binom{2k}{2(k-l)} r_l(r,s)(2l)!.$$

$R_n(r,s)$ is a polynomial of degree n in r and a monic polynomial of degree $2n$ in s. To prove the equality (11) it suffices to show that $R_k(r,s)$ satisfies (11).

Proposition 11. *For every positive integer n,*

$$(38)\quad R_n(r+2,s) - R_n(r,s) = (2n)^2 R_{n-1}(r,s).$$

Proof. Since linear combinations of $R_i(r+2,s)$ and $R_j(r,s)$ are expressed as ones of $r_l(r,s), l \in \mathbf{Z}$, we call a term containing $r_l(r,s)$ an l-th term of them. By (34) l-th terms of $R_n(r+2,s) - R_n(r,s)$ consist of a part of l-th terms of $R_n(r+2,s)$, a part of l-th terms of of $R_n(r,s)$ and a part of $(l-1)$-th terms of $R_n(r+2,s)$, and therefore the sum of them is equal to

$$(39)\quad \frac{1}{(2n)!}\left(2^{2n}n!\right)\frac{1}{l!2^{2l}}\left(\prod_{j=0}^{n-l-1}(1+2j)\right)^2 \left(\prod_{j=0}^{n-l-1}\left(-n+\frac{1}{2}+\frac{r-s}{2}+j\right)\right) \binom{2n}{2(n-l)}(2l)!\Bigg(\Bigg(\left(\frac{1}{2}+\frac{r-s}{2}-l\right) \Big/ \left(-n+\frac{1}{2}+\frac{r-s}{2}\right)-1\Bigg) - \frac{2(n-l)(1+2l)}{(-1+2n-2l)(-1+2n-r+s)}\Bigg) = \frac{1}{(2n-2)!}\left(2^{2n-2}(n-1)!\right) \frac{1}{l!2^{2l}}\left(\prod_{j=0}^{n-l-2}(1+2j)\right)^2 \left(\prod_{j=0}^{n-l-2}\left(-n+1+\frac{1}{2}+\frac{r-s}{2}+j\right)\right) \binom{2n-2}{2(n-l)-2}(2l)!\left(4n^2\right).$$

□

By (38) we can express $R_n(r,s)$ as a linear combination of $R_k(0,s)$, i.e.,

$$(40)\quad R_n(r,s) = \begin{pmatrix} r^n & r^{n-1} & \dots & 1 \end{pmatrix} \begin{pmatrix} (2n)^n & (2n)^{n-1} & \dots \\ (2n-2)^n & (2n-2)^{n-1} & \dots \\ \vdots & \vdots & \end{pmatrix}^{-1} \begin{pmatrix} \binom{n}{0} & \binom{n}{1} & \dots \\ \binom{n-1}{0} & \binom{n-1}{1} & \dots \\ \vdots & \vdots & \end{pmatrix} \begin{pmatrix} 4^0(n!/n!)^2 & 0 & \dots \\ 0 & 4^1(n!/(n-1)!)^2 & \\ \vdots & & \ddots \end{pmatrix} {}^t(R_n(0,s) \quad R_{n-1}(0,s) \quad \dots \quad R_0(0,s)).$$

Therefore by (40) we can reduce (11) to the following recurrence relation for $R_n(0,s)$:

$$(41)\quad aR_{n-2}(0,s) + bR_{n-1}(0,s) + cR_n(0,s) + 2(n+1)R_{n+1}(0,s) = 0$$

where we put

$$a = 8(-1+n)^2 n(-1+2n)(3+2n)(-6n+4n^2+3s-s^2),$$
$$b = 4n^2(-1+2n)(-5+14n+12n^2) - 2n(-3+8n+8n^2)(-3s+s^2),$$
$$c = 2(2+2n+22n^2+12n^3) - 2(1+n)(-3s+s^2).$$

We denote by $F(s)$ the left hand side of (41). The equality (41) is verified by evaluating the special values of $F(s)$ at the points $s=-2x+1$, 0 and $2x+2$ such that $0 \le x \le n$ and $x \in \mathbf{Z}$. We can compute such special values $F(s)$ by the next three propositions and have $F(s)=0$ at $2n+3$ different points. Since $F(s)$ is a polynomial in s of degree at most $2n+2$, $F(s)$ is identically zero. There still remain three propositions to be proven.

Proposition 12. *For every integer n, x such that $0 \le x \le n$,*

$$(42)\quad R_n(0,-2x+1) = (-4)^{(n-x)}(n!/x!)^2(2x)!.$$

Proof. It follows from (31), (37) that for integers n, m such that $0 \le m \le n$,

$$(43)\quad R_n(0,1-2m) = \sum_{j=m}^{n} \binom{2j}{j+m}\binom{n}{j}(n-m)! (-1)^{n-m}(2(n-j))!(j+m)!/((n-j)!) = (2m)!(-1)^{n-m}(-2(m-n))!\binom{n}{m} {}_2F_1(m+1/2, m-n; m-n+1/2; 1).$$

Thus Gauss' formula gives the desired result. □

In order to prove the remaining propositions we give the following lemma which is easily shown by (30), (31):

Lemma 1. *For positive integer n and non negative integer k,*

$$(44)\quad r_n(0,2k) = (-1)^{n+k}((2n-1)!!)^2 - 2\sum_{i=0}^{k-1}(-1)^{n+k}((2n-1)!!)^2 \prod_{j=1}^{i}(-2j-2n+1) \Big/ \prod_{j=1}^{i}(2j-2n-1).$$

Proposition 13. *For every positive integer n,*

$$(45)\quad R_n(0,0) = (-1)^n(2n)!,$$
$$(46)\quad R_n(0,2) = (-4)^n n!.$$

Proof. By using $\binom{n}{l} = \binom{n-1}{l} + \binom{n-1}{l-1}$ we have

$$(47)\quad R_n(0,0) = \sum_{l=0}^{n-1}(-n+1/2)(4(n-l)-2)((2(n-l)-3)!!)2^{n-l-1}\binom{n-1}{l}\left(\prod_{i=0}^{n-l-2}(-n+3/2+i)\right)r_l(0,0) + \sum_{l=0}^{n-1}(-n+1/2)(-1/2-l)^{-1}((2(n-l)-3)!!)2^{n-l-1}\binom{n-1}{l}\left(\prod_{i=0}^{n-l-2}(-n+3/2+i)\right)r_{l+1}(0,0) = -4n(n-1/2)R_{n-1}(0,0).$$

Therefore we get (45) by induction. On the other hand we have

$$(48)\quad R_n(0,2) = 2(1+2n)((-1+2n)!!)^2(-1)^{n+1} {}_3F_2(-n, 1/2, 1/2; 1/2-n, 3/2; 1)$$

by definition and therefore Saalschütz's theorem gives (46) (See first paragraph of Bailey [1, 4.3.]). □

Proposition 14. *For every integer n, x such that $0 \le x \le n$,*

$$R_n(0, 2+2x) = (-4)^{(n-x)}(n!/x!)^2(2x)!. \tag{49}$$

For the proof of this proposition we need the following two lemmas:

Lemma 2. *For non negative m and positive k,*

$$\begin{aligned} &-(-1-2k+2m-4km)R_{k-1}(0,m+1) \\ &+2(1+2k-2m)Q_{k-1}(m+1) \\ &+j_0(k,m)+j_c(k,m) = R_k(0,m), \end{aligned} \tag{50}$$

$$\begin{aligned} &-(1+2k-2m)(2m+1)R_{k-1}(0,m+1) \\ &-2(1+2k-2m)Q_{k-1}(m+1) \\ &= -(1+2k-2m)(-2k+2-(2m+1)) \\ &R_{k-1}(0,m) - (1+2k-2m) \\ &(-2k+2-(2m+1))j_0(k-1,m), \end{aligned} \tag{51}$$

$$\begin{aligned} &j_0(k,m) \\ &= (-2k+1-2m)(2k-2m)j_0(k-1,m), \end{aligned} \tag{52}$$

$$\begin{aligned} &j_c(k,m) \\ &= -(1+2m)(-k+1/2-m)j_0(k-1,m) \end{aligned} \tag{53}$$

where

$$\begin{aligned} Q_k(m) &= \sum_{l=0}^{k}(2(k-l)-1)!!2^{k-l}\binom{k}{l} \\ &\left(\prod_{i=0}^{k-l-1}(-k+1/2-m+i)\right) \\ &lr_l(0,m)(-1)^m, \end{aligned} \tag{54}$$

$$\begin{aligned} j_0(k,m) &= \sum_{l=0}^{k}(2(k-l)-1)!!2^{k-l}\binom{k}{l} \\ &\left(\prod_{i=1}^{k-l}(-k-1/2-m+i)\right) \\ &(r_l(0,m+1)-r_l(0,m+1))(-1)^m, \end{aligned} \tag{55}$$

$$\begin{aligned} j_c(m) &= \sum_{l=0}^{k-1}(2(k-l)-3)!!2^{k-l-1} \\ &\binom{k-1}{l}\left(\prod_{i=0}^{k-l-2}(-k+1/2-m+i)\right) \\ &(-1+2l-2m)(1+2m) \\ &(r_l(0,m+1)-r_l(0,m+1))(-1)^m/2. \end{aligned} \tag{56}$$

Proof. We can prove (50), (51), (52) and (53) in the same way as for the proof of (45). □

Lemma 3. *For $0 < m < k$,*

$$\begin{aligned} &2m(-1+2m)R_{k-1}(0,m+1) \\ &= -(1+2k-2m)(-k+1-(2m+1)) \\ &R_{k-1}(0,m) + R_k(0,m). \end{aligned} \tag{57}$$

Proof. We have (57) by eliminating $Q_{k-1}(m+1)$ from (50), (51) because $j_0(k,m) = j_c(k,m) = 0$ for $0 < m < k$ by (52), (53). Proposition 14 follows from (57) by induction. □

4. Finally we prove Proposition 2. By using the formula for $h_0(L,\alpha,\beta) = h_{20}(L,\alpha,\beta)$ in the proof of Kaufhold [4, Hilfssatz 6] we have

$$\begin{aligned} M(b_0,r) &= -\int_0^\infty W((y/2)E)y^{r-2}dy \\ &= -2^{-s-1}\pi^{1/2-r}(1+r)^{-1}\Gamma(s-1) \\ &\Gamma((-2+r+s)/2)\Gamma((1+r-s)/2) \\ &= 2^{-s-4}\pi^{-1}\Gamma(s-1)M(a_0,r). \end{aligned} \tag{58}$$

We get Proposition 2 by comparing this with (36).

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