Invariant subspaces of certain sub Hilbert spaces of H^2

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Abstract: Recently Yousefi and Hesameddini [13] have obtained a characterization for shift invariant subspaces of a special class of Hilbert spaces contained in the Hardy space H^2. In the present note we settle an open problem posed by them in their paper. In fact, by discussing invariance under multiplication by finite Blaschke factors we prove a far more general result than the main result of [13]. We prove our results under much weaker assumptions than the assumptions of [13] and with a simpler proof.

Key words: Invariant subspaces; Blaschke factor; inner function; wold decomposition.

1. Introduction. Let H be a Hilbert space contained in the Hardy space H^2. The inner products on H and H^2 are denoted by $\langle \cdot, \cdot \rangle_H$ and $\langle \cdot, \cdot \rangle_{H^2}$ respectively. In [13] Yousefi and Hesameddini assume H to satisfy the following axioms:

A1. If there are four functions $f_1, f_2, g_1, g_2 \in H$ such that $\langle f_1, g_1 \rangle_H = \langle f_2, g_2 \rangle_H$, then we have $\langle f_1, g_1 \rangle_{H^2} = \langle f_2, g_2 \rangle_{H^2}$.

A2. If φ is any inner function; then $\varphi f \in H$ and $\langle \varphi f, \varphi g \rangle_H = \langle f, g \rangle_H$ for all $f, g \in H$

and study the subspaces of H that are invariant under the operator S which stands for the multiplication by the co-ordinate function z. We record the subspace characterization obtained by Yousefi and Hesameddini for reference:

Theorem 1.1 (Theorem 3, [13]). *Let H be a Hilbert space contained in H^2 satisfying axioms A1 and A2. Let M be a closed subspace of H that is invariant under the operator S. Further if the set of multipliers of H coincide with H^∞, then there exists a unique inner function φ such that $M = \varphi H$.*

As demonstrated in [13] (Corollary 6), the famous theorem of Beurling [1], stated below, follows as a corollary to the above result since any closed subspace of H^2 will trivially satisfy the various assumptions of the main result (Theorem 3) of [13].

Theorem 1.2 (Beurling). *Let M be a closed subspace of H^2 that is invariant under the operator S. Then there exists an inner function φ which is unique up to a constant of modulus 1, such that $M = \varphi H^2$.*

The authors of [13] end their paper with the following open problem: "*Is every Hilbert space H satisfying axioms A1 and A2 of the form φH^2 for some $\varphi \in H^\infty$, and there exists a constant k such that $\langle f, g \rangle_H = k\langle f, g \rangle_{H^2}$ for every $f, g \in H$*". In the present note we answer this question in the affirmative by obtaining a characterization similar to the main result of [13] under much weaker assumptions. In fact we obtain a generalization in two directions: for one we weaken the conditions in [13] by requiring $A2$ to be true for a single predetermined finite Blaschke factor instead of for all inner functions. Also, we drop the requirement of [13] (Theorem 3) that needs to assume the nature of the multipliers of H. In addition, our proof seems to be simpler as it does not rely on the techniques of Shapiro [10] and instead follows from the Wold decomposition and a known result of [12].

2. Terminology and preliminary results. Let $\mathbf{D}$ denote the open unit disk, and its boundary, the unit circle by $\mathbf{T}$. The Lebesgue space L^p on the unit circle is a collection of complex valued functions f on the unit circle such that $\int |f|^p dm$ is finite, where dm is the normalized Lebesgue measure on $\mathbf{T}$.

The Hardy space H^p is the following closed subspace of L^p:

$$\left\{ f \in L^p : \int f z^n dm = 0 \ \forall \ n \geq 1 \right\}.$$

For $1 \leq p < \infty$, H^p is a Banach space under the norm

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$$\|f\|_p = \left(\int |f|^p dm \right)^{\frac{1}{p}}.$$

However H^∞ is a Banach space under the norm

$$\|f\|_\infty = \inf\{K : m\{z \in T : |f(z)| > K\} = 0\}.$$

The Hardy space H^2 turns out to be a Hilbert space under the inner product

$$\langle f, g \rangle = \int f\overline{g} dm.$$

An inner function is any function φ in H^2 that satisfies $\langle \varphi f, \varphi g \rangle_{H^2} = \langle f, g \rangle_{H^2}$ for every $f, g \in H^2$. Equivalently, this means that $\|\varphi f\|_{H^2} = \|f\|_{H^2}$ for every $f \in H^2$. By a finite Blaschke product $B(z)$ we mean

$$\prod_{i=1}^{n} \frac{z - \alpha_i}{1 - \overline{\alpha_i} z}$$

where each $\alpha_i \in \mathbf{D}$; and $\alpha_i \neq \alpha_j$ for $i \neq j$. Each finite Blaschke factor is an inner function. The isometric operator of multiplication by B on H^2 shall be denoted by T_B. When $n = 1$ and $\alpha_1 = 0$ then $B(z) = z$. For the rest of this paper we shall assume that $\alpha_1 = 0$. This shall not cause any loss in generality due to the conformal invariance of H^2. By $H^2(B)$ we denote the closed linear span of $\{B^n : n = 0, 1, \ldots\}$ in H^2. A detailed account of H^p spaces can be found in Duren [2], Garnett [3], Hoffman [4] and Koosis [5].

We now record results that will be used in our derivations. Note that throughout this paper $B(z)$ shall denote an arbitrarily chosen but then fixed finite Blaschke factor with $B(0) = 0$. First we cite a result by Singh and Thukral [12] that generalizes the main theorem of [11] which is in fact a generalization of a well known theorem de Branges [9]. In turn, the theorem of de Branges is a generalization of Beurling's theorem. This theorem of de Branges has proved to be the starting point of the important theory of de Branges spaces, see [6–9] and the references contained therein.

The difference between the result of [11] and that of de Branges (the scalar version) is that de Branges assumes H to be contractively contained in H^2 whereas Singh and Singh impose no topological assumptions on the containment of H in H^2. We first state

Theorem 2.1 (Singh and Thukral, [12]). *Let H be a Hilbert space such that*

(i) *H is algebraically contained in H^2;*
(ii) *$T_B(H) \subset H$;*
(iii) *T_B acts as an isometry on H.*

Then

$$H = \varphi_1 H^2(B) \oplus \cdots \oplus \varphi_r H^2(B)$$

where $\varphi_j \in H^\infty$, $1 \leq j \leq r$, and $r \leq n$. Further

$$\|\varphi_1 f_1 + \cdots + \varphi_r f_r\|_H^2 = \|f_1\|_{H^2}^2 + \cdots + \|f_r\|_{H^2}^2$$

for each $\varphi_1 f_1 + \cdots + \varphi_r f_r \in H$.

In the simplest case when $B(z) = z$ the above result reduces to

Theorem 2.2 (Singh and Singh [11]). *Let M be a Hilbert space which is a vector subspace of H^2. Suppose $S(M) \subset M$ and S acts as an isometry on M. Then there exists $b \in H^\infty$ such that $M = bH^2$, and $\|bf\|_M = \|f\|_{H^2}$ for all $f \in H^2$.*

As observed in [12], any function $f \in H^2$ can be expressed uniquely as

$$f = e_{0,0} f_0 + e_{1,0} f_1 + \cdots + e_{n-1,0} f_{n-1}$$

where $f_j \in H^2(B)$, $0 \leq j \leq n-1$ and

$$e_{j,0} = \frac{\sqrt{1 - |\alpha_{j+1}|^2}}{1 - \overline{\alpha_{j+1}} z} \left(\prod_{k=1}^{j} \frac{z - \alpha_k}{1 - \overline{\alpha_k} z} \right)$$

for $0 \leq j \leq n-1$. To each $r-$ tuple $(\varphi_1, \ldots, \varphi_r)$ of functions in H^∞ we associate an $n \times r$ matrix called the $B-$ matrix of $(\varphi_1, \ldots, \varphi_r)$ defined by

$$A = (\varphi_{ij}),\ 0 \leq i \leq n-1, \text{ and } 1 \leq j \leq r$$

where

$$\varphi_j = \sum_{i=0}^{n-1} e_{i,0} \varphi_{ij},\ 1 \leq j \leq r$$

We say that A is $B-$ inner if

$$(\overline{\varphi_{ji}})(\varphi_{ij}) = (\delta_{st})$$

where $1 \leq s, t \leq r$, and δ_{st} is the Kronecker delta.

The following lemma proved in [12] gives a necessary and sufficient condition for a $B-$ matrix to be $B-$ inner.

Lemma 2.3. *The $B-$ matrix of the $r-$ tuple $(\varphi_1, \ldots, \varphi_r)$ of H^∞ functions is $B-$ inner if and only if $\{B^m \varphi_i : 1 \leq i \leq r, m = 0, 1, 2, \ldots\}$ is an orthonormal set in H^2.*

Corollary 2.4. *An H^∞ function φ is inner if and only if $\{z^m \varphi : m = 0, 1, 2, \ldots\}$ is an orthonormal set.*

3. Main results.

Theorem 3.1. *Let H be a Hilbert space such that:*

(i) *H is a vector subspace of H^2;*
(ii) *$T_B(H) \subset H$ and $\langle Bf, Bg\rangle_H = \langle f, g\rangle_H$ for all $f, g \in H$;*
(iii) *If $f_1, f_2, g_1, g_2 \in H$ satisfy $\langle f_1, g_1\rangle_H = \langle f_2, g_2\rangle_H$, then we have $\langle f_1, g_1\rangle_{H^2} = \langle f_2, g_2\rangle_{H^2}$.*

Then there exist H^∞ functions $b_1, b_2, \ldots, b_r (r \le n)$ such that

$$H = b_1H^2(B) \oplus \cdots \oplus b_rH^2(B)$$

and the $B-$ matrix of the $r-$ tuple $(b_1, \ldots, b_r)$ is $B-$ inner.

Proof. In view of (ii), we see that the operator T_B acts isometrically on H. In fact T_B and H satisfy the conditions of Theorem 2.1 and thus there exist H^∞ functions $b_1, b_2, \ldots, b_r (r \le n)$ such that

$$H = b_1H^2(B) \oplus \cdots \oplus b_rH^2(B)$$

and

$$\|b_1f_1 + \cdots + b_rf_r\|_H^2 = \|f_1\|_{H^2}^2 + \cdots + \|f_r\|_{H^2}^2$$

for all $f_1, \ldots, f_r \in H^2(B)$.

Further from the proof of Theorem 2.1, as given in [12], we observe that the isometric operator T_B acts as a shift on H, and thus H has the Wold type decomposition:

$$H = W \oplus T_B(W) \oplus T_B^2(W) \oplus \cdots \tag{1}$$

Here W is the wandering subspace $H \ominus T_B(H)$ which turns out to have dimension r, $r \le n$, and $b_1, \ldots, b_r$ are the orthonormal vectors (in H) spanning W.

From equation (1) note that for each $j = 1, \ldots, r$, $\langle b_jB^n, b_jB^m\rangle_H = 0$ whenever $n \ne m$. This implies $\langle b_jB^n, b_jB^m\rangle_{H^2} = 0$ whenever $n \ne m$ by assumption (iii). Thus

$$\{b_jB^n : n = 0, 1, \ldots\}$$

is orthogonal in H^2. So the collection

$$\left\{\frac{b_j}{\|b_j\|_{H^2}} B^n : n = 0, 1, \ldots\right\}$$

is orthonormal in H^2 for all $j = 1, \ldots, r$.
Now

$$\begin{aligned} H &= b_1H^2(B) \oplus \cdots \oplus b_rH^2(B) \\ &= \left(\frac{b_1}{\|b_1\|_{H^2}}\right)H^2(B) \oplus \cdots \oplus \left(\frac{b_r}{\|b_r\|_{H^2}}\right)H^2(B). \end{aligned}$$

To see that the $B-$matrix of the $r-$tuple $\left(\frac{b_1}{\|b_1\|_{H^2}}, \ldots, \frac{b_r}{\|b_r\|_{H^2}}\right)$ is $B-$ inner observe that the set

$$\left\{\frac{b_j}{\|b_j\|_{H^2}} B^n : 1 \le j \le r, n = 0, 1, \ldots\right\}$$

is orthonormal in H^2. The desired claim follows once again from Lemma 2.3. □

Corollary 3.2. *Let H be a Hilbert space such that*:
(i) *H is a vector subspace of H^2;*
(ii) *$S(H) \subset H$ and $\langle zf, zg\rangle_H = \langle f, g\rangle_H$ for all $f, g \in H$;*
(iii) *If $f_1, f_2, g_1, g_2 \in H$ satisfy $\langle f_1, g_1\rangle_H = \langle f_2, g_2\rangle_H$, then we have $\langle f_1, g_1\rangle_{H^2} = \langle f_2, g_2\rangle_{H^2}$.*

Then there exists a unique inner function b such that $H = bH^2$ and there is a constant k such that $\|bf\|_H = k\|f\|_{H^2}$ for all $f \in H^2$.

Proof. Take $B(z) = z$ in Theorem 3.1. □

Theorem 3.3. *Let H be a Hilbert space satisfying the conditions of Corollary 3.2. Let M be a closed subspace of H which is invariant under S. Then there exists a unique inner function φ such that $M = \varphi H$.*

Proof. By Corollary 3.2, there exists an inner function b such that $H = bH^2$. Since M is a closed subspace of H, so M is also a Hilbert space satisfying the assumptions of Corollary 3.2. Thus there exists an inner function c such that $M = cH^2$. Note that $c = b\varphi$ for some $\varphi \in H^2$. This implies that $|\varphi| = 1$ *a.e.* Further, $M = \varphi bH^2 = \varphi H$. □

Clearly Theorem 1.1 comes as a corollary to the above Theorem 3.3.

4. Solution to the open problem in [13]. The paper [13] ends with the following problem: "*Is every Hilbert space H satisfying axioms $A1$ and $A2$ of the form φH^2 for some $\varphi \in H^\infty$, and there exists a constant k such that $\langle f, g\rangle_H = k\langle f, g\rangle_{H^2}$ for every $f, g \in H$*". Let us explain how Corollary 3.2 answers this open problem in the afirmative. Assume that H is a Hilbert space contained in H^2 and it satisfies axioms $A1$ and $A2$ stated in section 1. Clearly, H satisfies the conditions of Corollary 3.2. So there exists an inner function φ and a constant k such that $H = \varphi H^2$ and $\|\varphi f\|_H = k\|f\|_{H^2}$ for each $f \in H^2$.

In view of the polarization law we see that:

$$\begin{aligned}4\langle \varphi f, \varphi g\rangle_H &= \|\varphi f + \varphi g\|_H^2 - \|\varphi f - \varphi g\|_H^2 \\ &\quad + i\|\varphi f + i\varphi g\|_H^2 - i\|\varphi f - i\varphi g\|_H^2 \\ &= k^2\|f+g\|_{H^2}^2 - k^2\|f-g\|_{H^2}^2 \\ &\quad + ik^2\|f+ig\|_{H^2}^2 - ik^2\|f-ig\|_{H^2}^2 \\ &= 4k^2\langle f, g\rangle_{H^2} \\ &= 4k^2\langle \varphi f, \varphi g\rangle_{H^2}.\end{aligned}$$

Thus the inner product on H satisfies the condition as stated in the problem.

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