Kirchhoff elastic rods in higher-dimensional space forms

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Abstract: In this paper, we give examples of Kirchhoff rod centerlines fully immersed in higher-dimensional space forms. More precisely, we prove that any helix in a space form is a Kirchhoff rod centerline. These examples imply the difference of the geometric properties between Kirchhoff rod centerlines and elasticae.

Key words: Elastica; Kirchhoff elastic rod; helix; variational problem.

1. Introduction. The *elastica* and the *Kirchhoff elastic rod* (or simply *Kirchhoff rod*) are both classical mathematical models of thin elastic rods. The elastica is probably the simplest model, and is characterized as a critical curve of the energy of bending only. On the other hand, the Kirchhoff rod is some more complicated model, and is characterized as a critical framed curve of the energy with the effects of both bending and twisting. The curve obtained by eliminating the frame of a Kirchhoff rod is called a *Kirchhoff rod centerline*. Then a Kirchhoff rod centerline is a generalization of an elastica.

These curves were originally considered in the two or three-dimensional Euclidean space, but these notions (or their generalizations) are naturally extended to those in general Riemannian manifolds (see, e.g., [1,2,8,9,12–14,16,17,20]).

In this paper, we consider Kirchhoff rod centerlines in simply-connected n-dimensional space forms $\mathcal{M} = \mathbf{R}^n, S^n, H^n$, $n \geq 2$. It is known that when $n = 2, 3$, all Kirchhoff rod centerlines in $\mathcal{M}$ are explicitly expressed in terms of Jacobi sn function and the elliptic integrals ([11], see also [6,10,15,18,19,21,22]). However, in the case where $n \geq 4$, examples of Kirchhoff rod centerlines fully immersed in $\mathcal{M}$ are not known. The purpose of this paper is to give examples of Kirchhoff rod centerlines fully immersed in $\mathcal{M} = \mathbf{R}^n, S^n, H^n$, where $n \geq 4$. We obtain the following main theorem.

Theorem 1.1 (Theorem 4.1). *Let γ be any helix in $\mathcal{M} = \mathbf{R}^n, S^n, H^n$, where $n \geq 2$. Then γ is a Kirchhoff rod centerline.*

Here, a helix is defined to be a curve all of whose Frenet curvatures are constant. (For details about the definition of a helix, see Section 3.) Since there exists a helix in $\mathcal{M}$ which does not lie in any $(n-1)$-dimensional totally geodesic submanifold of $\mathcal{M}$, we have the following

Corollary 1.2 (Corollary 4.2). *There exists a Kirchhoff rod centerline in $\mathcal{M} = \mathbf{R}^n, S^n, H^n$, $n \geq 2$ which does not lie in any $(n-1)$-dimensional totally geodesic submanifold of $\mathcal{M}$.*

On the other hand, as for elasticae, the following result is known ([16], see also [5]).

Proposition 1.3 (Langer-Singer [16]). *Let γ be an elastica in $\mathcal{M} = \mathbf{R}^n, S^n, H^n$, where $n \geq 4$. Then γ lies in a three-dimensional totally geodesic submanifold of $\mathcal{M}$.*

Corollary 1.2 and Proposition 1.3 show the difference of the geometric properties between elasticae and Kirchhoff rod centerlines in space forms.

2. Elasticae and Kirchhoff rod centerlines. In this section, we define an elastica and a Kirchhoff rod centerline.

Let $\mathcal{M}$ be $\mathbf{R}^n$, S^n, H^n of constant sectional curvature G. We denote by $\langle *, * \rangle$ the Riemannian metric of $\mathcal{M}$, and by $|*|$ the norm. Unless otherwise specified, all curves, vector fields etc. are assumed to be of class C^∞.

First we define an elastica. Let $\gamma = \gamma(t) : [t_1, t_2] \to \mathcal{M}$ be a unit-speed curve in $\mathcal{M}$. We denote by $T(t) = \gamma'(t)$ the tangent vector to γ and by $\nabla_T = \nabla_{\partial/\partial t}$ the covariant derivative along γ. The bending energy of γ is defined to be the total squared curvature of γ, that is,

$$\mathfrak{F}(\gamma) = \int_{t_1}^{t_2} |\nabla_T T|^2 dt.$$

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We call γ an elastica if γ is a critical point of the bending energy $\mathfrak{F}$ with respect to the variations of unit-speed curves which preserve the end points $\gamma(t_1), \gamma(t_2)$ and the tangent vectors $T(t_1), T(t_2)$ at the end points. More precisely, an elastica is defined to be a solution of the associated Euler-Lagrange equation:

$$(1) \quad \nabla_T[2(\nabla_T)^2T + (3|\nabla_T T|^2 - \mu + 2G)T] = 0,$$

where μ is a real constant. For the derivation of the Euler-Lagrange equation in a general Riemannian manifold, see Section 1 of [16].

Definition 2.1. A unit-speed curve γ in $\mathcal{M}$ is called an *elastica* if there exists $\mu \in \mathbf{R}$ such that (1) holds.

Next we define a Kirchhoff rod, which is a mathematical model of an elastic rod with the effects of both bending and twisting. The twisting of an elastic rod cannot be represented by a curve γ only. (Note that the torsion τ or the higher order Frenet curvatures of γ are not directly related to the twisting of the elastic rod.) To describe how the elastic rod is twisted, we utilize an orthonormal frame field $M = (M_1, M_2, \ldots, M_{n-1})$ in the normal bundle $T^\perp\mathcal{M}$ along γ. We call such a pair $\{\gamma, M\}$ a *unit-speed curve with adapted orthonormal frame*, and γ the *centerline* of $\{\gamma, M\}$.

Let ν be a fixed positive constant, which is determined by the material of the elastic rod. We define the energy $\mathfrak{T}$, which includes the effects of both bending and twisting, as follows:

$$\mathfrak{T}(\{\gamma, M\}) = \mathfrak{F}(\gamma) + \nu \sum_{i=1}^{n-1} \int_{t_1}^{t_2} |\nabla_T^\perp M_i|^2 dt,$$

where $\nabla^\perp$ denotes the normal connection in $T^\perp\mathcal{M}$, so that, $\nabla_T^\perp M_i = \nabla_T M_i - \langle \nabla_T M_i, T\rangle T$. Here, the first term of $\mathfrak{T}(\{\gamma, M\})$ expresses the energy of bending, and the second term that of twisting. We call $\{\gamma, M\}$ a Kirchhoff rod if $\{\gamma, M\}$ is a critical point of $\mathfrak{T}$ with respect to the variations of unit-speed curves with adapted orthonormal frames which preserve the end points $\gamma(t_1)$, $\gamma(t_2)$ and the orthonormal frames $(T(t_1), M(t_1))$, $(T(t_2), M(t_2))$ at the end points. More precisely, a Kirchhoff rod is defined to be a solution of the associated Euler-Lagrange equations:

$$(2) \quad \nabla_T\Bigg[2(\nabla_T)^2T + \bigg(3|\nabla_T T|^2 - \mu + 2G + \nu\sum_{i=1}^{n-1}|\nabla_T^\perp M_i|^2\bigg)T - 4\nu\sum_{i=1}^{n-1}\langle\nabla_T T, M_i\rangle \nabla_T^\perp M_i\Bigg] = 0,$$

$$(3) \quad (\nabla_T^\perp M_1, \ldots, \nabla_T^\perp M_{n-1}) = (M_1, \ldots, M_{n-1})a,$$

where $\mu \in \mathbf{R}$ and $a \in \mathfrak{o}(n-1)$. Here, $\mathfrak{o}(n-1)$ stands for the Lie algebra of all skew-symmetric matrices of size $n-1$. For the derivation of the Euler-Lagrange equation in a general Riemannian manifold, see Section 2 of [10].

Definition 2.2. Let $\{\gamma, M\}$ be a unit-speed curve with adapted orthonormal frame in $\mathcal{M}$. We call $\{\gamma, M\}$ a *Kirchhoff rod* if there exist $\mu \in \mathbf{R}$ and $a \in \mathfrak{o}(n-1)$ such that (2) and (3) hold. The matrix a is uniquely determined, and is called the *twist matrix* of $\{\gamma, M\}$.

We note that the matrix a in (3) is independent of t. Therefore, we see that if $\{\gamma, M\}$ is a Kirchhoff rod, then the integrand of the second term of $\mathfrak{T}(\{\gamma, M\})$ is independent of t. Physically, this means that the twist of an elastic rod in equilibrium is uniformly distributed along the centerline.

Remark 2.3. Let $\varphi \in O(n-1)$, where $O(n-1)$ stands for the Lie group of all orthogonal matrices of size $n-1$. A straightforward calculation using (2) and (3) yields that if $\{\gamma, M\}$ is a Kirchhoff rod, then $\{\gamma, M\varphi\}$ is also a Kirchhoff rod. Note that if the twist matrix of $\{\gamma, M\}$ is a, then that of $\{\gamma, M\varphi\}$ is $\varphi^{-1}a\varphi$.

We define a Kirchhoff rod centerline as follows:

Definition 2.4. A unit-speed curve γ in $\mathcal{M}$ is called a *Kirchhoff rod centerline* if there exists an orthonormal frame field $M = (M_1, M_2, \ldots, M_{n-1})$ in the normal bundle along γ such that $\{\gamma, M\}$ is a Kirchhoff rod.

By (1), (2) and (3), we see the following

Proposition 2.5. *Let γ be an elastica in $\mathcal{M}$. We take an orthonormal frame $M^0 = (M_1^0, \ldots, M_{n-1}^0)$ of the normal vector space at a point $\gamma(t_0)$ on the curve γ. Let $M = (M_1, \ldots, M_{n-1})$ be the parallel translation of M^0 with respect to the normal connection along γ. Then $\{\gamma, M\}$ is a Kirchhoff rod with twist matrix 0. Therefore, γ is a Kirchhoff rod centerline. Conversely, if $\{\gamma, M\}$ is a Kirchhoff rod with twist matrix 0, then γ is an elastica and M is parallel with respect to the normal connection.*

The above proposition implies that a Kirchhoff rod centerline is a generalization of an elastica.

3. Helices. In this section, we recapitulate the Frenet frame and Frenet curvatures of a curve in a space form and define a helix.

Let $\mathcal{M} = \mathbf{R}^n$, S^n, H^n, and let d be an integer satisfying $2 \le d \le n$. A unit-speed curve γ in $\mathcal{M}$ is called a *Frenet curve of osculating rank* d if T, $\nabla_T T, \dots, (\nabla_T)^{d-1}T$ are linearly independent for each t, and T, $\nabla_T T, \dots, (\nabla_T)^{d-1}T$, $(\nabla_T)^d T$ are linearly dependent for each t. For a Frenet curve γ of osculating rank d, let $(N_0, N_1, \dots, N_{d-1})$ be the orthonormal d-frame along γ obtained by applying the Gram-Schmidt orthogonalization to $(T, \nabla_T T, (\nabla_T)^2 T, \dots, (\nabla_T)^{d-1}T)$, and let $\kappa_1(t), \dots, \kappa_{d-1}(t)$ be the functions defined by $\kappa_i = \langle \nabla_T N_{i-1}, N_i \rangle$, $i = 1, \dots, d-1$. Then $\kappa_1, \dots, \kappa_{d-1}$ are positive functions and the following Frenet formula holds:

$$(\nabla_T N_0, \dots, \nabla_T N_{d-1}) = (N_0, \dots, N_{d-1}) f, \tag{4}$$

where

$$f = \begin{pmatrix} 0 & -\kappa_1 & & & & \mathbf{0} \\ \kappa_1 & 0 & -\kappa_2 & & & \\ & \kappa_2 & 0 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & 0 & -\kappa_{d-1} \\ \mathbf{0} & & & & \kappa_{d-1} & 0 \end{pmatrix}.$$

The orthonormal d-frame $(N_0, \dots, N_{d-1})$ is called the *Frenet d-frame along* γ, and the function κ_i is called the *i-th Frenet curvature* of γ.

By a similar argument to that in the case of $\mathbf{R}^3$ [4], we can check the following holds. Given arbitrary $d-1$ positive functions $\kappa_1, \dots, \kappa_{d-1}$, there exists a curve γ of osculating rank d whose i-th Frenet curvature coincides with κ_i for $i = 1, \dots, d-1$. Such γ is uniquely determined up to isometries of $\mathcal{M}$. We can also check that a Frenet curve of osculating rank d lies in a d-dimensional totally geodesic submanifold of $\mathcal{M}$, and does not lie in any $(d-1)$-dimensional totally geodesic submanifold of $\mathcal{M}$.

A Frenet curve γ of osculating rank d is called a *helix of order* d if all the Frenet curvatures $\kappa_1, \dots, \kappa_{d-1}$ are constant functions. Also, a unit-speed curve γ is called a *helix* if γ is a helix of order d for some d.

4. Main theorem. In this section, we express the Euler-Lagrange equation (2) in terms of the Frenet frame, and state the main theorem.

Let $\mathcal{M} = \mathbf{R}^n$, S^n, H^n, where $n \ge 4$. Let $\{\gamma, M\}$ be a unit-speed curve with adapted orthonormal frame in $\mathcal{M}$ defined on $I = [t_1, t_2]$. Suppose that γ is a Frenet curve of osculating rank n, and let $(N_0, \dots, N_{n-1})$ denote the Frenet n-frame along γ, and $\kappa_1, \dots, \kappa_{n-1}$ the Frenet curvatures of γ. We fix $t_0 \in I$. By Remark 2.3, we may assume, without loss of generality, that $M(t_0) = (N_1(t_0), \dots, N_{n-1}(t_0))$.

Suppose that (3) holds for some $a \in \mathfrak{o}(n-1)$. It follows from (4) that

$$\begin{aligned} &\nabla_T \left[2(\nabla_T)^2 T + \left(3|\nabla_T T|^2 - \mu + 2G \right.\right. \\ &\qquad \left.\left. + \nu \sum_{i=1}^{n-1} |\nabla_T^\perp M_i|^2 \right) T \right] \\ &\quad = \left[2\kappa_1'' + (\kappa_1)^3 + \left(-\mu + 2G + \nu \sum_{i,j=1}^{n-1} (a^j{}_i)^2 \right) \kappa_1 \right. \\ &\qquad \left. - 2\kappa_1(\kappa_2)^2 \right] N_1 + (4\kappa_1'\kappa_2 + 2\kappa_1\kappa_2') N_2 \\ &\qquad + 2\kappa_1\kappa_2\kappa_3 N_3, \end{aligned}$$

where $'$ denotes the differentiation with respect to t, and $a^j{}_i$ the (j,i)-component of a.

To calculate $\nabla_T[\sum_{i=1}^{n-1} \langle \nabla_T T, M_i \rangle \nabla_T^\perp M_i]$, we also use a $\nabla^\perp$-parallel frame along γ. Let $P = (P_1, P_2, \dots, P_{n-1})$ denote the parallel translation of $(N_1(t_0), \dots, N_{n-1}(t_0))$ with respect to the normal connection $\nabla^\perp$ along γ. Let $w_i = \langle \nabla_T T, M_i \rangle$ and $k_i = \langle \nabla_T T, P_i \rangle$, $i = 1, 2, \dots, n-1$. Then

$$\begin{aligned} &(\nabla_T T, \nabla_T M_1, \dots, \nabla_T M_{n-1}) \\ &\quad = (T, M_1, \dots, M_{n-1}) \begin{pmatrix} 0 & -{}^t\boldsymbol{w} \\ \boldsymbol{w} & a \end{pmatrix}, \\ &(\nabla_T T, \nabla_T P_1, \dots, \nabla_T P_{n-1}) \\ &\quad = (T, P_1, \dots, P_{n-1}) \begin{pmatrix} 0 & -{}^t\boldsymbol{k} \\ \boldsymbol{k} & 0 \end{pmatrix}, \end{aligned}$$

where $\boldsymbol{w} = {}^t(w_1, \dots, w_{n-1})$ and $\boldsymbol{k} = {}^t(k_1, \dots, k_{n-1})$. The orthonormal frames $(T, M_1, \dots, M_{n-1})$ and $(T, N_1, \dots, N_{n-1})$ are expressed as

$$\begin{aligned} &(T, M_1, \dots, M_{n-1}) \\ &\quad = (T, P_1, \dots, P_{n-1}) \begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix}, \\ &(T, N_1, \dots, N_{n-1}) \\ &\quad = (T, P_1, \dots, P_{n-1}) \begin{pmatrix} 1 & 0 \\ 0 & \psi \end{pmatrix}, \end{aligned}$$

for some $\xi, \psi : I \to O(n-1)$. A straightforward calculation yields

$$\begin{aligned} &\boldsymbol{w} = {}^t\xi \boldsymbol{k} = \xi^{-1}\boldsymbol{k}, \quad a = \xi^{-1}\xi', \\ &{}^t(\kappa_1, 0, \dots, 0) = {}^t\psi \boldsymbol{k} = \psi^{-1}\boldsymbol{k}, \quad b = \psi^{-1}\psi', \end{aligned} \tag{5}$$

where $b : I \to \mathfrak{o}(n-1)$ is defined by

$$b = \begin{pmatrix} 0 & -\kappa_2 & & & & \mathbf{0} \\ \kappa_2 & 0 & -\kappa_3 & & & \\ & \kappa_3 & 0 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & 0 & -\kappa_{n-1} \\ \mathbf{0} & & & & \kappa_{n-1} & 0 \end{pmatrix}.$$

Therefore, ξ, ψ are the solutions of the following initial value problems, respectively:

$$\xi'(t) = \xi(t)a, \quad \xi(t_0) = e, \tag{6}$$

$$\psi'(t) = \psi(t)b(t), \quad \psi(t_0) = e, \tag{7}$$

where e stands for the identity matrix of size $n-1$. The solution of (6) is explicitly expressed as $\xi(t) = \exp[(t-t_0)a]$. By using (5), we see

$$\nabla_T \left[\sum_{i=1}^{n-1} \langle \nabla_T T, M_i \rangle \nabla_T^{\perp} M_i \right] = \nabla_T [(P_1, \ldots, P_{n-1}) a \boldsymbol{k}] = (P_1, \ldots, P_{n-1}) a \boldsymbol{k}' = (N_1, \ldots, N_{n-1}) \psi^{-1} a \psi \begin{pmatrix} \kappa_1' \\ \kappa_1 \kappa_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Therefore, (2) is expressed in terms of the Frenet frame as follows:

$$\left[2\kappa_1'' + (\kappa_1)^3 + \left(-\mu + 2G + \nu \sum_{i,j=1}^{n-1} (a^j{}_i)^2 \right) \kappa_1 - 2\kappa_1 (\kappa_2)^2 \right] N_1 + (4\kappa_1' \kappa_2 + 2\kappa_1 \kappa_2') N_2 + 2\kappa_1 \kappa_2 \kappa_3 N_3 - 4\nu (N_1, \ldots, N_{n-1}) \psi^{-1} a \psi \begin{pmatrix} \kappa_1' \\ \kappa_1 \kappa_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0. \tag{8}$$

Before stating the main theorem, we describe the case where $n = 2, 3$. First, let $n = 3$. Then since $O(2)$ is commutative, $\psi^{-1} a \psi = a$ holds, and the equation (8) reduces to

$$\left[2\kappa_1'' + (\kappa_1)^3 + (-\mu + 2G + 2\nu (a^2{}_1)^2) \kappa_1 - 2\kappa_1 \kappa_2 (\kappa_2 - 2\nu a^2{}_1) \right] N_1 + (4\kappa_1' \kappa_2 + 2\kappa_1 \kappa_2' - 4\nu a^2{}_1 \kappa_1') N_2 = 0. \tag{9}$$

It is known that all the solutions κ_1, κ_2 of (9) are explicitly expressed by Jacobi sn function. Moreover, γ themselves are explicitly expressed by Jacobi sn function and the elliptic integrals in terms of cylindrical coordinates, and various properties of Kirchhoff rods are investigated ([6,10,11,19,21], etc.). Next let $n = 2$. Then for any unit-speed curve $\{\gamma, M_1\}$ with adapted orthonormal frame, $\nabla_T^{\perp} M_1 = 0$ holds. Thus (3) is satisfied for $a = 0$. Therefore, Proposition 2.5 yields that $\{\gamma, M_1\}$ is a Kirchhoff rod if and only if γ is an elastica. The equation (8) reduces to

$$\left[2\kappa_1'' + (\kappa_1)^3 + (-\mu + 2G) \kappa_1 \right] N_1 = 0. \tag{10}$$

The solutions κ_1 of (10) are also explicitly expressed by Jacobi sn function, and elasticae in $\mathcal{M} = \mathbf{R}^2$, S^2, H^2 are extensively studied ([3,7,15,16,20], etc.).

We return now to the case of $n \geq 4$. In this case, it seems to be difficult to obtain the explicit expressions of all the solutions of (8). But, we can construct some examples of Kirchhoff rod centerlines. The following main theorem holds.

Theorem 4.1. *Let γ be any helix in $\mathcal{M} = \mathbf{R}^n, S^n, H^n$, where $n \geq 2$. Then γ is a Kirchhoff rod centerline.*

Proof. First, we consider the case where γ is a helix of order n. Let $(N_0, \ldots, N_{n-1})$ denote the Frenet n-frame, and $\kappa_1, \ldots, \kappa_{n-1}$ (> 0) the Frenet curvatures of γ. We fix a point t_0, and let P, ψ, b be as above, and let $M = P\exp[(t-t_0)a]$, where $a \in \mathfrak{o}(n-1)$. Then (3) holds. Also, since b is a constant matrix, the solution ψ of (7) is explicitly expressed as $\psi = \exp[(t-t_0)b]$.

Let $n \geq 4$. We seek for a and μ satisfying (8). If we assume $[a, b] = 0$, then $\psi^{-1} a \psi = a$ holds, and hence the equation (8) reduces to the following:

$$(\kappa_1)^2 - \mu + 2G + \nu \sum_{i,j=1}^{n-1} (a^j{}_i)^2 - 2(\kappa_2)^2 - 4\nu a^1{}_2 \kappa_2 = 0, \tag{11}$$

$$\kappa_3 - 2\nu a^3{}_2 = 0, \tag{12}$$

$$a^4{}_2 = a^5{}_2 = \cdots = a^{n-1}{}_2 = 0. \tag{13}$$

And so we set

$$a = \frac{1}{2\nu} b,$$

$$\mu = (\kappa_1)^2 + \frac{1}{2\nu} \sum_{i=2}^{n-1} (\kappa_i)^2 + 2G.$$

Then we see that $[a, b] = 0$, and (11), (12) and (13) hold. Thus $\{\gamma, M\}$ is a Kirchhoff rod, and so γ is a Kirchhoff rod centerline.

Next let $n = 3$. In this case, it immediately follows that there exists $\mu \in \mathbf{R}$ satisfying (9). Thus $\{\gamma, M\}$ is a Kirchhoff rod, and hence γ is a Kirchhoff rod centerline. In the case of $n = 2$, by setting $\mu = \kappa_1^2 + 2G$, the equation (10) is satisfied, and hence γ is a Kirchhoff rod centerline.

We consider the case where γ is a helix of order $2 \leq d \leq n-1$. Then γ lies in a d-dimensional totally geodesic submanifold $\mathcal{N}$ of $\mathcal{M}$. By an argument similar to the above, we see that there exists an orthonormal $(d-1)$-frame field $(M_1, \ldots, M_{d-1})$ along γ such that $\{\gamma, (M_1, \ldots, M_{d-1})\}$ is a Kirchhoff rod in $\mathcal{N}$. Since $\mathcal{N}$ is a totally geodesic submanifold, we can take an orthonormal $(n-d)$-frame field $(M_d, \ldots, M_{n-1})$ along γ such that $\nabla_T M_d = \cdots = \nabla_T M_{n-1} = 0$ and $(M_d(t), \ldots, M_{n-1}(t))$ is an orthonormal basis of the normal vector space $T^{\perp}_{\gamma(t)}\mathcal{N}$ for each t. Then we can check that $\{\gamma, (M_1, \ldots, M_{d-1}, M_d, \ldots, M_{n-1})\}$ is a Kirchhoff rod in $\mathcal{M}$. Hence γ is a Kirchhoff rod centerline in $\mathcal{M}$. $\square$

For arbitrary positive numbers $\kappa_1, \ldots, \kappa_{n-1}$, there exists a helix γ of order n whose i-th Frenet curvature coincides with κ_i for $i = 1, \ldots, n-1$. Since γ does not lie in any $(n-1)$-dimensional totally geodesic submanifold of $\mathcal{M}$, we obtain

Corollary 4.2. *There exists a Kirchhoff rod centerline in $\mathcal{M} = \mathbf{R}^n, S^n, H^n$, $n \geq 2$ which does not lie in any $(n-1)$-dimensional totally geodesic submanifold of $\mathcal{M}$.*

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