An exponential Diophantine equation related to powers of two consecutive Fibonacci numbers

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Abstract: Here, we show that there is no integer $s \geq 3$ such that the sum of sth powers of two consecutive Fibonacci numbers is a Fibonacci number.

Key words: Fibonacci numbers; Applications of linear forms in logarithms.

1. Introduction. Let $(F_m)_{m\geq 1}$ be the Fibonacci sequence given by $F_1 = 1, F_2 = 1$ and $F_{m+2} = F_{m+1} + F_m$ for all $m \geq 1$. It is well-known that $F_m^2 + F_{m+1}^2 = F_{2m+1}$. Hence, $F_m^s + F_{m+1}^s$ is a Fibonacci number for all $m \geq 1$ when $s \in \{1, 2\}$. In the paper [4], the following result was proved:

Theorem A. *If $s \geq 1$ is an integer such that $F_m^s + F_{m+1}^s$ is a Fibonacci number for all sufficiently large m, then $s \in \{1, 2\}$.*

Note that this doesn't say much about the Diophantine equation

$$F_m^s + F_{m+1}^s = F_n \tag{1}$$

in integers $m \geq 1$, $n \geq 1$, $s \geq 3$. It merely says that if $s \geq 3$ is fixed, then there are infinitely many positive integers m for which $F_m^s + F_{m+1}^s$ is not a Fibonacci number. Thus, the main result from [4] cannot answer to the question of whether the Diophantine equation (1) has finitely or infinitely many integer solutions (m, n), even for fixed $s \geq 3$. Let us briefly describe the proof of this result from [4]. Put $\alpha := (1 + \sqrt{5})/2$, $\beta := (1 - \sqrt{5})/2$. Then it is known that

$$F_m = \frac{\alpha^m - \beta^m}{\sqrt{5}} \quad \text{holds for all} \quad m \geq 1. \tag{2}$$

Hence, for fixed s we have that

$$F_m^s = \left(\frac{\alpha^m - \beta^m}{\sqrt{5}}\right)^s = \frac{\alpha^{ms}}{5^{s/2}} + O_s\left(\alpha^{m(s-2)}\right).$$

The constant implied by the above O depends on s. Applying the above estimate with m and $m + 1$ and adding them up we get

$$\begin{aligned} F_m^s + F_{m+1}^s &= \frac{\alpha^{ms} + \alpha^{(m+1)s}}{5^{s/2}} + O_s(\alpha^{m(s-2)}) \\ &= \frac{\alpha^{ms}(1 + \alpha^s)}{5^{s/2}} + O_s(\alpha^{m(s-2)}). \end{aligned}$$

Comparing the above estimate with F_n given by (2) we get right away that

$$\frac{\alpha^{ms-n}(1 + \alpha^s)}{5^{(s-1)/2}} - 1 = O_s(\alpha^{m(s-2)-n} + \alpha^{-2n}),$$

and the right–hand side above is of smaller order of magnitude than α^{ms-n} for large m and n. Hence, the left hand side must be zero for large m, so we must have $n = ms + t$, where t is some fixed integer such that

$$1 + \alpha^s = 5^{(s-1)/2}\alpha^t. \tag{3}$$

From here, the authors of [4] proceed to prove that the above equation (3) has no integer solutions (s, t) with $s \geq 3$ by using linear forms in logarithms and calculations. We note in passing that there is no need for linear forms in logarithms in order to solve (3). Indeed, conjugating it and multiplying the two relations we get

$$(1 + \alpha^s)(1 + \beta^s) = (-1)^{s-1}5^{s-1}(\alpha\beta)^t = (-1)^{s-1-t}5^{s-1}.$$

The left–hand side is positive and smaller than $2(1 + \alpha^s) < 2(1 + 2^s)$. Hence, we get

$$2(1 + 2^s) > 5^{s-1},$$

which has no solution for any $s \geq 3$ anyway.

Our main result is the following.

Theorem 1. *Equation* (1) *has no solutions* (m, n, s) *with* $m \geq 2$ *and* $s \geq 3$.

2. The proof of Theorem 1.

2.1. The plan of attack. Our method is roughly as follows. We use linear forms in loga-

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rithms together with the observation that F_m^s is smaller by an exponential factor in s than F_{m+1}^s to deduce some inequality for s versus m and n. When m is small, say $m \leq 150$, we use continued fractions to lower the bounds and then brute force to cover the range of the potential solutions. When $m > 150$, our bound on s versus m and n says that F_m^s can be sufficiently well approximated by $\alpha^{ms}/5^{s/2}$ and similarly F_{m+1}^s can be sufficiently well approximated by $\alpha^{(m+1)s}/5^{s/2}$. A further application of the linear forms in logarithms together with some computations finish the job.

Now let's get to work.

2.2. An inequality for s in terms of m and n. Observe that when $m = 1$ we have $F_m^s + F_{m+1}^s = 2 = F_3$ for all $s \geq 1$, which is why we imposed that $m \geq 2$.

When $m = 2$, we get that $F_n = 1 + F_3^s = 1 + 2^s$ which has no integer solutions (n, s) with $s \geq 3$ (see [2] for a list of all solutions to the Diophantine equation $F_n = 1 + x^s$ in positive integers (n, s, x) with $s \geq 2$).

From now on, we assume that $m \geq 3$. Since $s \geq 3$, we get that $F_n \geq F_3^3 + F_4^3$, so that $n \geq 10$. Using formula (2), we rewrite equation (1) as

$$\frac{\alpha^n}{\sqrt{5}} - F_{m+1}^s = F_m^s + \frac{\beta^n}{\sqrt{5}}. \tag{4}$$

The right–hand side above is a number in the interval $[F_m^s - 1, F_m^s + 1]$. In particular, it is positive. Dividing both sides of equation (4) by F_{m+1}^s, we get

$$\left|\alpha^n 5^{-1/2} F_{m+1}^{-s} - 1\right| < 2\left(\frac{F_m}{F_{m+1}}\right)^s < \frac{2}{1.5^s}, \tag{5}$$

where we used the fact that $F_m/F_{m+1} \leq 2/3$ for all $m \geq 2$. This last inequality is equivalent to the inequality $2F_{m+1} \geq 3F_m$. Replacing the number F_{m+1} by $F_m + F_{m-1}$, the last inequality above is seen to be equivalent to $2F_{m-1} \geq F_m = F_{m-1} + F_{m-2}$, which is equivalent to $F_{m-1} \geq F_{m-2}$, which is true for all $m \geq 2$.

We shall use several times a result of Matveev (see [5] or Theorem 9.4 in [1]), which asserts that if $\alpha_1, \alpha_2, \ldots, \alpha_k$ are positive real algebraic numbers in an algebraic number field $\mathbf{K}$ of degree D, b_1, $b_2, \ldots, b_k$ are rational integers, and

$$\Lambda := \alpha_1^{b_1}\alpha_2^{b_2}\cdots\alpha_k^{b_k} - 1$$

is not zero, then

$$|\Lambda| > \exp\bigl(-C_{k,D}(1 + \log B)A_1 \cdots A_k\bigr) \tag{6}$$

where

$$C_{k;D} := 1.4 \times 30^{k+3} \times k^{4.5} \times D^2(1 + \log D),$$
$$B \geq \max\{|b_1|, |b_2|, \ldots, |b_k|\},$$

and

$$A_i \geq \max\{Dh(\alpha_i), |\log \alpha_i|, 0.16\}, \quad i = 1, 2, \ldots, k.$$

Here, for an algebraic number η we write $h(\eta)$ for its logarithmic height whose formula is

$$h(\eta) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\Bigl(\max\{|\eta^{(i)}|, 1\}\Bigr)\right),$$

with d being the degree of η over $\mathbf{Q}$ and

$$f(X) := a_0 \prod_{i=1}^{d}(X - \eta^{(i)}) \in \mathbf{Z}[X]$$

being the minimal primitive polynomial over the integers having positive leading coefficient and η as a root.

In a first application of Matveev's theorem, we take $k := 3$, $\alpha_1 := \alpha$, $\alpha_2 := \sqrt{5}$, $\alpha_3 := F_{m+1}$. We also take $b_1 := n$, $b_2 := -1$, $b_3 := -s$. We thus take

$$\Lambda_1 := \alpha^n 5^{-1/2} F_{m+1}^{-s} - 1. \tag{7}$$

The absolute value of Λ_1 appears in the left–hand side of inequality (5). To see that $\Lambda_1 \neq 0$, observe that imposing that $\Lambda_1 = 0$ yields $\alpha^n = \sqrt{5}F_{m+1}^s$, so $\alpha^{2n} \in \mathbf{Z}$, which is false for positive n. Thus, $\Lambda_1 \neq 0$.

The algebraic number field containing $\alpha_1, \alpha_2, \alpha_3$ is $\mathbf{K} := \mathbf{Q}(\sqrt{5})$ which is quadratic, so we can take the degree $D := 2$. Since the height of α_1 satisfies $h(\alpha_1) = (\log \alpha)/2 = 0.240606\ldots$, it follows that we can take $A_1 := 0.5 > 2h(\alpha_1)$. Since the height of α_2 satisfies $h(\alpha_2) = (\log 5)/2 = 0.804719\ldots$, it follows that we can take $A_2 := 1.61$. Since the inequality $F_\ell < \alpha^{\ell-1}$ holds for all integers $\ell \geq 1$, it follows that $h(\alpha_3) = \log F_{m+1} < m \log \alpha$, therefore we can take $A_3 := 2m \log \alpha$. Finally, observe that

$$\alpha^{n-1} > F_n = F_m^s + F_{m+1}^s \geq F_{m+1}^s > (\alpha^{m-1})^s,$$
$$\alpha^{n-2} < F_n = F_m^s + F_{m+1}^s < (F_m + F_{m+1})^s = F_{m+2}^s < \alpha^{(m+1)s}.$$

In particular, $n \geq (m-1)s \geq 2s$, so we can take $B := n$. It is also the case that

$$B = n \leq (m+1)s + 1 < (m+2)s.$$

Applying inequality (6) to get a lower bound for $|\Lambda_1|$ and comparing this with inequality (5), we get

$$\exp\big(-C_{3,2}(1+\log n)\times 0.5\times 1.61\times 2m\log\alpha\big) < \frac{2}{1.5^s},$$

where

$$C_{3,2} = 1.4\times 30^6\times 3^{4.5}\times 2^2\times(1+\log 2) < 10^{12}.$$

Hence, we get

$$\begin{aligned} s &< \frac{\log 2}{\log 1.5} + 10^{12}\times 0.5\times 1.61\times(2\log\alpha)\\ &\quad\times(\log 1.5)^{-1}m(1+\log n)\\ &< 2\times 10^{12}m(1+\log n) < 3\times 10^{12}m\log n, \end{aligned}$$

where we also used the fact that $\log n \geq \log 10 > 2$. Together with the fact that $n < (m+2)s$, we get that

$$s < 3\times 10^{12}m\log((m+2)s). \tag{8}$$

2.3. The case of small m. We next treat the cases when $m\in[3,150]$. In this case,

$$s < 3\times 10^{12}m\log((m+2)s) \leq 4.5\times 10^{14}\log(152s),$$

giving $s < 1.92\times 10^{16}$. Thus,

$$n < (m+2)s \leq 152s < 3\times 10^{18}.$$

We next take another look at Λ_1 given by expression (7). Put

$$\Gamma_1 := n\log\alpha - \log\sqrt{5} - s\log F_{m+1}.$$

Thus, $\Lambda_1 = e^{\Gamma_1}-1$.

Observe that Γ_1 is positive since Λ_1 is positive because the right–hand side of equation (4) is positive. Thus,

$$0 < \Gamma_1 < e^{\Gamma_1}-1 = \Lambda_1 < \frac{2}{1.5^s},$$

so

$$\begin{aligned} 0 &< n\left(\frac{\log\alpha}{\log F_{m+1}}\right) - s - \left(\frac{\log\sqrt{5}}{\log F_{m+1}}\right)\\ &< \frac{2}{1.5^s\log(F_{m+1})} < \frac{2}{1.5^s} < \frac{2}{(1.5^{\frac{1}{152}})^n}. \end{aligned} \tag{9}$$

We now apply the following result due to Dujella and Pethő [3].

Lemma 2. *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational γ such that $q > 6M$, and let μ be some real number. Let $\varepsilon := \|\mu q\| - M\|\gamma q\|$. If $\varepsilon > 0$, then there is no solution to the inequality*

$$0 < n\gamma - s + \mu < AB^{-n}$$

in positive integers n and s with

$$\frac{\log(Aq/\varepsilon)}{\log B} \leq n \leq M.$$

For us, inequality (9) is

$$0 < n\gamma - s + \mu < AB^{-n},$$

where

$$\begin{aligned} \gamma &:= \frac{\log\alpha}{\log F_{m+1}}, & \mu &:= -\frac{\log\sqrt{5}}{\log F_{m+1}},\\ A &:= 2, & B &:= 1.0026 < 1.5^{\frac{1}{152}}. \end{aligned}$$

We take $M := 3\times 10^{18}$. For each of our numbers m, we take $q := q_{99}$ to be the denominator of the 99th convergent to γ. Note that q depends on m. The minimal value of q for $m\in[3,150]$ exceeds the number $10^{45} > 6M$. Thus, we may apply Lemma 2 for each such q, γ and μ. The maximal value of $M\|q\gamma\|$ computed is smaller than 10^{-28}, whereas the minimal value of $\|q\mu\|$ is $> 1.1\times 10^{-17}$. Thus, we can take $\varepsilon := \|q\mu\| - M\|q\gamma\| > 10^{-17}$. Also, the maximal value of q is $< 2\times 10^{62}$. Hence, by Lemma 2, all solutions (n,s) of inequality (9) have

$$\begin{aligned} n &< \frac{\log(Aq/\varepsilon)}{\log B} < \frac{\log(2\times(2\times 10^{62})\times 10^{17})}{\log(1.0026)}\\ &< 71000. \end{aligned}$$

Next, since $(m-1)s \leq n$, we have

$$s \leq n/(m-1) < 71000/(m-1).$$

A computer search with Mathematica revealed in less than one hour that there are no solutions to the equation (1) in the range $m\in[3,150]$, $n\in[10,71000]$ and $s\in[3,71000/(m-1)]$. This completes the analysis in the case $m\in[3,150]$.

2.4. An upper bound on s in terms of m. From now on, we assume that $m\geq 151$. Recall that, by (8), we have

$$s < 3\times 10^{12}m\log((m+2)s). \tag{10}$$

Next we give an upper bound on s depending only on m. If

$$s \leq m+2, \tag{11}$$

then we are through. Otherwise, that is if $m+2 < s$, we then have

$$s < 3 \times 10^{12} m \log(s^2) = 6 \times 10^{12} m \log s,$$

which can be rewritten as

$$\frac{s}{\log s} < 6 \times 10^{12} m. \tag{12}$$

Since the function $x \mapsto x/\log x$ is increasing for all $x > e$, it is easy to check that the inequality

$$\frac{x}{\log x} < A \quad \text{yields} \quad x < 2A \log A,$$

whenever $A \geq 3$. Indeed, for if not, then we would have that $x > 2A \log A > e$, therefore

$$\frac{x}{\log x} > \frac{2A \log A}{\log(2A \log A)} > A,$$

where the last inequality follows because $2 \log A < A$ holds for all $A \geq 3$. This is a contradiction. Taking $A := 6 \times 10^{12} m$ in the above argument, we get that inequality (12) implies that

$$\begin{aligned} s &< 2(6 \times 10^{12} m) \log(6 \times 10^{12} m) \\ &= 12 \times 10^{12} m (\log(6 \times 10^{12}) + \log m) \\ &< 12 \times 10^{12} m (30 + \log m) \\ &< 12 \times 10^{12} m \times (7 \log m) \\ &< 10^{14} m \log m. \end{aligned} \tag{13}$$

In the above inequalities, we used the fact that $\log m \geq \log 151 > 5$. From (11) and (13), we conclude that the inequality

$$s < 10^{14} m \log m \tag{14}$$

holds for all $m \geq 151$.

2.5. An absolute upper bound on *s*. Let us look at the element

$$x := \frac{s}{\alpha^{2m}}.$$

From the above inequality (14), it follows that

$$x < \frac{10^{14} m \log m}{\alpha^{2m}} < \frac{1}{\alpha^m}, \tag{15}$$

where the last inequality holds for all $m \geq 80$. In particular, $x < \alpha^{-151} < 10^{-31}$. We now write

$$F_m^s = \frac{\alpha^{ms}}{5^{s/2}} \left(1 - \frac{(-1)^m}{\alpha^{2m}}\right)^s,$$

$$F_{m+1}^s = \frac{\alpha^{(m+1)s}}{5^{s/2}} \left(1 - \frac{(-1)^{m+1}}{\alpha^{2(m+1)}}\right)^s.$$

If m is odd, then

$$\begin{aligned} 1 < \left(1 - \frac{(-1)^m}{\alpha^{2m}}\right)^s &= \left(1 + \frac{1}{\alpha^{2m}}\right)^s < \mathrm{e}^x \\ &< 1 + 2x \end{aligned}$$

because $x < 10^{-31}$ is very small, while if m is even, then

$$\begin{aligned} 1 > \left(1 - \frac{(-1)^m}{\alpha^{2m}}\right)^s &= \exp\left(\log\left(1 - \frac{1}{\alpha^{2m}}\right) s\right) > \mathrm{e}^{-2x} \\ &> 1 - 2x, \end{aligned}$$

again because $x < 10^{-31}$ is very small. The same inequalities are true if we replace m by $m+1$. Thus, we have that

$$\max\left\{\left|F_m^s - \frac{\alpha^{ms}}{5^{s/2}}\right|, \left|F_{m+1}^s - \frac{\alpha^{(m+1)s}}{5^{s/2}}\right|\right\} < \frac{2x\alpha^{(m+1)s}}{5^{s/2}}. \tag{16}$$

We now return to our equation (1) and rewrite it as

$$\begin{aligned} \frac{\alpha^n - \beta^n}{5^{1/2}} = F_n = F_m^s + F_{m+1}^s &= \frac{\alpha^{ms}}{5^{s/2}} + \frac{\alpha^{(m+1)s}}{5^{s/2}} \\ &+ \left(F_m^s - \frac{\alpha^{ms}}{5^{s/2}}\right) + \left(F_{m+1}^s - \frac{\alpha^{(m+1)s}}{5^{s/2}}\right), \end{aligned}$$

or

$$\begin{aligned} &\left|\frac{\alpha^n}{5^{1/2}} - \frac{\alpha^{ms}}{5^{s/2}}(1 + \alpha^s)\right| \\ &= \left|\frac{\beta^n}{5^{1/2}} + \left(F_m^s - \frac{\alpha^{ms}}{5^{s/2}}\right) + \left(F_{m+1}^s - \frac{\alpha^{(m+1)s}}{5^{s/2}}\right)\right| \\ &< \frac{1}{\alpha^n} + \left|F_m^s - \frac{\alpha^{ms}}{5^{s/2}}\right| + \left|F_{m+1}^s - \frac{\alpha^{(m+1)s}}{5^{s/2}}\right| \\ &< \frac{1}{\alpha^n} + 2x\left(\frac{\alpha^{ms}(1 + \alpha^s)}{5^{s/2}}\right). \end{aligned}$$

Thus, dividing both sides by $\alpha^{(m+1)s}/5^{s/2}$, we conclude that

$$\begin{aligned} &|\alpha^{n-(m+1)s} 5^{(s-1)/2} - (1 + \alpha^{-s})| \\ &< \frac{5^{s/2}}{\alpha^{n+(m+1)s}} + 2x(1 + \alpha^{-s}) < \frac{1}{2\alpha^m} + \frac{5x}{2} \\ &< \frac{3}{\alpha^m}, \end{aligned} \tag{17}$$

where we used the fact that

$$\frac{5^{s/2}}{\alpha^{(m+1)s}} \leq \left(\frac{\sqrt{5}}{\alpha^{152}}\right)^s < \frac{1}{2}, \quad n \geq (m-1)s > m,$$

and $\alpha^s \geq \alpha^3 > 4$, as well as inequality (15). Hence, we conclude that

$$\left|\alpha^{n-(m+1)s} 5^{(s-1)/2} - 1\right| < \frac{1}{\alpha^s} + \frac{3}{\alpha^m} \leq \frac{4}{\alpha^\ell}, \tag{18}$$

where we put $\ell := \min\{m, s\}$. We now set

$$\Lambda_2 := \alpha^{n-(m+1)s}5^{(s-1)/2} - 1. \tag{19}$$

and observe that $\Lambda_2 \neq 0$. Indeed, for if $\Lambda_2 = 0$, then $\alpha^{2((m+1)s-n)} = 5^{s-1} \in \mathbf{Z}$, which is possible only when $(m+1)s = n$. But if this were so, then we would get $0 = \Lambda_2 = 5^{(s-1)/2} - 1$, which leads to the conclusion that $s = 1$, which is not possible. Hence, $\Lambda_2 \neq 0$. Next, let us notice that since $s \geq 3$ and $m \geq 151$, we have that

$$|\Lambda_2| \leq \frac{1}{\alpha^3} + \frac{1}{\alpha^{151}} < \frac{1}{2}, \tag{20}$$

so that $\alpha^{n-(m+1)s}5^{(s-1)/2} \in [1/2, 2]$. In particular,

$$\begin{aligned}(m+1)s - n &< \frac{1}{\log\alpha}\left(\frac{(s-1)\log 5}{2} + \log 2\right)\\ &< s\left(\frac{\log 5}{2\log\alpha}\right) < 1.7s;\\ (m+1)s - n &> \frac{1}{\log\alpha}\left(\frac{(s-1)\log 5}{2} - \log 2\right)\\ &> 1.6s - 4. \end{aligned} \tag{21}$$

It follows from (21) that $(m+1)s - n > 1.6s - 4 > 0$ because $s \geq 3$. We lower bound the left–hand side of inequality (18) using again Matveev's theorem. We take $k := 2, \alpha_1 := \alpha, \alpha_2 := \sqrt{5}$. We also take the exponents $b_1 := n - (m+1)s$ and $b_2 := s - 1$. As in the previous application of Matveev's result, we can take $D := 2$, $A_1 := 0.5, A_2 := 1.61$. Also, we can take $B := 1.7s > \max\{|b_1|, |b_2|\}$ by inequality (21). We thus get that

$$\exp(-C_{2,2}(1 + \log(1.7s)) \times 0.5 \times 1.61) < \frac{4}{\alpha^\ell},$$

where

$$C_{2,2} = 1.4 \times 30^5 \times 2^{4.5} \times 2^2 \times (1 + \log 2) < 5.3 \times 10^9.$$

This leads to

$$\begin{aligned}\ell &< \frac{\log 4}{\log\alpha} + 5.3 \times 10^9 \times 0.5 \times 1.61 \times (\log\alpha)^{-1}\\ &\quad \times (1 + \log(1.7s))\\ &< 9 \times 10^9(1.6 + \log s) < 9 \times 2.6 \times 10^9 \log s\\ &< 3 \times 10^{10}\log s.\end{aligned}$$

Here, we used the fact that $1 + \log(1.7) < 1.6$. If $\ell = s$, we then get that $s < 3 \times 10^{10}\log s$, so

$$s < 10^{12}.$$

If $\ell = m$, then using also (14), we get that

$$m < 3 \times 10^{10}\log s < 3 \times 10^{10}\log\bigl(10^{14}m\log m\bigr).$$

The last inequality above leads to $m < 2 \times 10^{12}$, so, by (14) once again, we get that

$$s < 10^{14} \times (2 \times 10^{12})\log(2 \times 10^{12}) < 6 \times 10^{27}.$$

So, at any rate, we have that

$$s < 6 \times 10^{27}. \tag{22}$$

2.6. A better upper bound on s. Next, we take $\Gamma_2 := (s-1)\log\sqrt{5} - ((m+1)s - n)\log\alpha$. Observe that $\Lambda_2 = e^{\Gamma_2} - 1$, where Λ_2 is given by (19). Since $|\Lambda_2| < 1/2$ (see inequality (20)), we have that $e^{|\Gamma_2|} < 2$. Hence,

$$|\Gamma_2| \leq e^{|\Gamma_2|}|e^{\Gamma_2} - 1| < 2|\Lambda_2| < \frac{2}{\alpha^s} + \frac{6}{\alpha^m}.$$

This leads to

$$\left|\frac{\log\sqrt{5}}{\log\alpha} - \frac{(m+1)s - n}{s-1}\right| < \frac{1}{(s-1)\log\alpha}\left(\frac{2}{\alpha^s} + \frac{6}{\alpha^m}\right). \tag{23}$$

Note first that $\alpha^m \geq \alpha^{151} > 3 \times 10^{31} > 10^3 s$ by estimate (22).

Assume next that $s > 100$. Then $\alpha^s > 1000s$. Hence, we get that

$$\begin{aligned}\frac{1}{(s-1)\log\alpha}\left(\frac{2}{\alpha^s} + \frac{6}{\alpha^m}\right) &< \frac{8}{s(s-1)1000\log\alpha}\\ &< \frac{1}{60(s-1)^2}.\end{aligned} \tag{24}$$

Estimates (23) and (24) lead to

$$\left|\frac{\log\sqrt{5}}{\log\alpha} - \frac{(m+1)s - n}{s-1}\right| < \frac{1}{60(s-1)^2}. \tag{25}$$

By a criterion of Legendre, inequality (25) implies that the rational number $((m+1)s - n)/(s-1)$ is a convergent to $\gamma := (\log\sqrt{5})/(\log\alpha)$. Let $[a_0, a_1, a_2, a_3, a_4, \ldots] = [1, 1, 2, 19, 2, 9, \ldots]$ be the continued fraction of γ, and let p_k/q_k be its kth convergent. Assume that $((m+1)s - n)/(s-1) = p_t/q_t$ for some t. Then $s - 1 = dq_t$ for some positive integer d, which in fact is the greatest common divisor of $(m+1)s - n$ and $s - 1$. We have the inequality $q_{54} > 1.4 \times 10^{28} > s - 1$. Thus, $t \in \{0, \ldots, 53\}$. Furthermore, $a_k \leq 29$ for all $k = 0, 1, \ldots, 53$. From the known properties of continued fractions, we have that

$$\begin{aligned}\left|\gamma - \frac{(m+1)s - n}{s-1}\right| = \left|\gamma - \frac{p_t}{q_t}\right| &> \frac{1}{(a_t + 2)q_t^2}\\ &\geq \frac{d^2}{31(s-1)^2} \geq \frac{1}{31(s-1)^2},\end{aligned}$$

which contradicts inequality (25). Hence, $s \leq 100$.

2.7. The final step. To finish, we go back to inequality (17) and rewrite it as

$$\left|\alpha^{n-(m+1)s}5^{(s-1)/2}(1+\alpha^{-s})^{-1}-1\right| < \frac{3}{\alpha^m(1+\alpha^{-s})} < \frac{3}{\alpha^m}.$$

Recall that $s \in [3, 100]$ and

$$1.6s - 4 < (m+1)s - n < 1.7s.$$

Put $t := (m+1)s - n$. We computed all the numbers $|\alpha^{-t}5^{(s-1)/2}(1+\alpha^{-s})^{-1}-1|$ for all $s \in [3, 100]$ and all $t \in [\lfloor 1.6s-4 \rfloor, \lfloor 1.7s \rfloor]$. None of them ended up being zero, since if it were we would get the Diophantine equation $\alpha^s + 1 = 5^{(s-1)/2}\alpha^{s-t}$, which has no integer solution (s,t) with $s \geq 3$ by the arguments from the beginning. The smallest of these numbers is $> 3/10^3$. Thus, $3/10^3 < 3/\alpha^m$, or $\alpha^m < 10^3$, so $m < 15$, which is false.

The theorem is therefore proved.

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