# Another proof on the existence of Mertens's constant

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**Abstract:** Let $\mathcal{P}$ denote the set of all prime numbers, and let $p_k$ denothe the $k$th prime. In 1873 Mertens presented a quantitative proof of the divergence of the series $\sum_{p\in\mathcal{P}} \frac{1}{p}$ by showing the limit $B := \lim_{x\to\infty}(\sum_{p\le x} \frac{1}{p} - \log\log x)$ exists with $B = 0.26149\ldots$. In this paper we give another proof on the divergence of the above series. We prove the following

**Theorem.** *The sequence* $f(n) := \sum_{k=1}^{n} \frac{1}{p_k} - \log\log n$, $n = 2, 3, \ldots,$ *is decreasing and bounded from below, and its limit equals the Mertens's constant* $B$.

In proofs of the first two conditions we use only classical estimations for $p_k$, obtained in 1939 and 1962 by Rosser and Schoenfeld.

**Key words:** Mertens's constant; prime number.

**1. Introduction.** In what follows, we use the notation from the Abstract. Moreover, $\Delta(x)$ denotes the error term $|\sum_{p\le x} \frac{1}{p} - \log\log x - B|$, and $\gamma$ denotes the Euler constant.

We shall prove below the Theorem stated in the Abstract. But before that, a few words are in order about former proofs.

In 1997 Lindqvist and Peetre [4, Remarks 2 and 4 in Section 1] listed known proofs of Mertens's theorem [5] and noticed that all of them depended on the asymptotic result [3, Theorem 425]:

$$\sum_{p\le x} \frac{\log p}{p} = \log x + O(1) \text{ as } x \to \infty \tag{1}$$

(see, e.g., the proof of Theorem 427 in [3]). In [4, Sections 3 and 4] they estimated the error term of the expansion $H := \sum_{p\in\mathcal{P}}\big(\log\big(1 - \frac{1}{p}\big) + \frac{1}{p}\big)$; this, by the equality $B = \gamma + H$, allowed them to compute $B$ with 200 decimals.

A deeper analysis of Mertens's result and its proofs is presented in the paper of 2005 by Villarino [9]. Additionally, he recalls the estimations of $\Delta(x)$ obtained earlier:

(a) $\Delta(x) < \frac{1}{10\log^2 x} + \frac{4}{15\log^3 x}$, for $x \ge 10372$ (1998, Dusart [1, Theorem 11]),
what improved the result of 1962 by Rosser and Schoenfeld [7, Theorem 5 and its Corollary] that $\Delta(x) < \frac{1}{\log^2 x}$, for $x > 1$;

(b) $\Delta(x) < \frac{3\log x + 4}{8\pi\sqrt{x}}$, for $x \ge 13.5$ (1976, Schoenfeld [8]; under the Riemann Hypothesis).

From the form of $\Delta(x)$ it immediately follows that this function behaves somewhat chaotically: there are very long intervals $[a, b]$ on which the sum $s(x) := \sum_{p\le x} \frac{1}{p}$ is constant; and $s$ increases "more rapidly" on $[p_m, p_{m+1}]$ than $\log\log x$ if $p_m, p_{m+1}$ are twin primes. Probably for this reason the use of the estimate (1) in classical proofs is indispensable.

**2. Proof of the Theorem.** In the proof we shall apply the following estimations of $p_k$ obtained by Rosser [6], and Rosser and Schoenfeld [7, Corollary to Theorem 3], respectively:

$$p_k > k\log k, \text{ for } k \ge 1, \tag{R}$$

$$p_k < k(\log k + \log\log k), \text{ for } k \ge 6. \tag{RS}$$

(Sharper bounds for $p_k$ were obtained by Rosser and Schoenfeld [7], and Dusart [1,2], but we do not use them here.)

We shall prove now the sequence $(f(n))_{n=2}^{\infty}$ is decreasing. Indeed, by inequality $(R)$, the form of $f$ and the inequality $\log(1 + \frac{1}{t}) > \frac{1}{t+1}$ for $t > 0$, we obtain

$$f(n+1) - f(n) = \frac{1}{p_{n+1}} - \log\left(1 + \frac{\log(1 + \frac{1}{n})}{\log n}\right) < \frac{1}{p_{n+1}} - \log\left(1 + \frac{1}{(n+1)\log n}\right) < \frac{1}{p_{n+1}} - \frac{1}{(n+1)\log n + 1} <$$

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$$\frac{(n+1)\log n + 1 - (n+1)\log(n+1)}{M_n} =$$
$$\frac{1-(n+1)\log(1+\frac{1}{n})}{M_n} < \frac{1-(n+1)\cdot\frac{1}{n+1}}{M_n} = 0,$$

where $M_n = p_{n+1}\cdot((n+1)\log n + 1)$.

For the proof that the sequence $(f(n))_{n=2}^{\infty}$ is bounded from below we shall need the auxiliary inequality

$$f(k+1) - f(k) > \frac{1}{p_{k+1}} - \frac{1}{p_k} - \frac{\log\log k}{k\log^2 k}, \tag{2}$$

for $k \geq 6$. Its proof is based on inequalies $\log(1+t) < t$, for $t > 0$, and $(RS)$: for $k \geq 6$ we have

$$f(k+1) - f(k) = \frac{1}{p_{k+1}} - (\log\log(k+1) - \log\log k) =$$
$$\frac{1}{p_{k+1}} - \log\left(\frac{\log(1+\frac{1}{k})}{\log k} + 1\right) >$$
$$\frac{1}{p_{k+1}} - \log\left(\frac{1}{k\log k} + 1\right) >$$
$$\frac{1}{p_{k+1}} - \frac{1}{p_k} + \frac{1}{p_k} - \frac{1}{k\log k} \overset{(RS)}{>}$$
$$\frac{1}{p_{k+1}} - \frac{1}{p_k} + \frac{1}{k(\log k + \log\log k)} - \frac{1}{k\log k} =$$
$$\frac{1}{p_{k+1}} - \frac{1}{p_k} - \frac{1}{k}\cdot\frac{\log\log k}{(\log k + \log\log k)\log k} >$$
$$\frac{1}{p_{k+1}} - \frac{1}{p_k} - \frac{\log\log k}{k\log^2 k}.$$

Now, by (2), for $n \geq 6$ we obtain

$$f(n+1) - f(6) = \sum_{k=6}^{n}(f(k+1) - f(k)) >$$
$$-\frac{1}{p_6} + \frac{1}{p_{n+1}} - \sum_{k=6}^{n}\frac{\log\log k}{k\log^2 k} >$$
$$-\frac{1}{p_6} - \int_5^n \frac{\log\log x}{x\log^2 x}\,\mathrm{d}x =$$
$$-\frac{1}{p_6} - \left(\frac{\log\log x}{\log x} + \frac{1}{\log x}\right)\Bigg|_5^n \searrow -0.9939\ldots$$

as $n \to \infty$. Hence, since $f(6) = 0.76082\ldots$, for $n \geq 7$ we obtain $f(n) > -1 + 0.76 = -0.24$.

To prove that $\lim_{n\to\infty} f(n) = B$, assume first that $n \geq 3$ (this is necessary in the integral below), set

$$B(x) = \sum_{p\leq x}\frac{1}{p} - \log\log x,$$

and notice that the summation is over primes $p_k$ with $k \leq \pi(x)$, the number of primes $\leq x$. Hence, by $(R)$, we obtain

$$0 < f(n) - B(n) = \sum_{k=\pi(n)+1}^{n}\frac{1}{p_k} < \tag{3}$$
$$\int_{\pi(n)}^{n}\frac{\mathrm{d}x}{x\log x} = \log\frac{\log n}{\log \pi(n)} =$$
$$\log\left(1 + \frac{\log\frac{n}{\pi(n)}}{\log\pi(n)}\right) < \frac{\log\frac{n}{\pi(n)}}{\log\pi(n)}.$$

Since $\pi(n) > \frac{n}{\log n}$ for $n \geq 17$ (see [7, Corollary 1 to Theorem 2]), the latter fraction in (3) is less than $\frac{\log\log n}{\log n - \log\log n}$ (for such $n$'s) and tends to 0 as $n \to \infty$. By the Mertens result, $\lim_{n\to\infty} f(n) = B$ $(= \lim_{n\to\infty} B(n))$.

The proof of the Theorem is complete. □

From the above proof we also obtain the following estimation (because $f(n)$ decreases to $B$):

$$0 < f(n) - B < \Delta(n) + \frac{\log\log n}{\log n - \log\log n} \quad \text{for } n \geq 17.$$

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