Norm estimates and integral kernel estimates for a bounded operator in Sobolev spaces

By Yoichi Miyazaki

School of Dentistry, Nihon University, 1-8-13 Kanda-Surugadai, Chiyoda-ku, Tokyo 101-8310, Japan

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Abstract: We show that a bounded linear operator from the Sobolev space $W_r^{-m}(\Omega)$ to $W_r^m(\Omega)$ is a bounded operator from $L_p(\Omega)$ to $L_q(\Omega)$, and estimate the operator norm, if $p, q, r \in [1, \infty]$ and a positive integer m satisfy certain conditions, where Ω is a domain in $\mathbf{R}^n$. We also deal with a bounded linear operator from $W_{p'}^{-m}(\Omega)$ to $W_p^m(\Omega)$ with $p' = p/(p-1)$, which has a bounded and continuous integral kernel. The results for these operators are applied to strongly elliptic operators.

Key words: Sobolev space; kernel theorem; Sobolev embedding theorem; elliptic operator.

1. Introduction. In [2,3] we developed the L_p theory for elliptic operators in divergence form subject to the Dirichlet boundary condition. Let A be the $2m$th-order elliptic operator

$$(1.1)\qquad Au(x) = \sum_{\substack{|\alpha|\le m\\ |\beta|\le m}} D^\alpha(a_{\alpha\beta}(x)D^\beta u(x)),$$
$$D = -\sqrt{-1}\partial$$

in a domain Ω of $\mathbf{R}^n$. One of the main results is that, for each $p \in (1,\infty)$, the inverse of $A-\lambda$ is a bounded linear operator

$$(A-\lambda)^{-1} : W_p^{-m}(\Omega) \to W_{p,0}^m(\Omega)$$

for λ in a suitable region of the complex plane $\mathbf{C}$, and that it satisfies

$$(1.2)\qquad \|(A-\lambda)^{-1}\|_{W_p^{-i}(\Omega)\to W_p^j(\Omega)} \le C|\lambda|^{-1+(i+j)/2m}$$

for $0 \le i \le m$, $0 \le j \le m$ with some constant C. We also derived estimates for the kernels of e^{-tA} and $(A-\lambda)^{-1}$, based on (1.2). However, we used (1.2) only for $i = 0$. The aim of this paper is to present two theorems which are useful for making a full use of (1.2) including the case $0 < i \le m$.

Throughout this paper, we assume that Ω is $\mathbf{R}^n$ or a uniform C^1 domain if $n \ge 2$, and that Ω is an interval in $\mathbf{R}$ if $n = 1$. For $p \in (1,\infty)$ and $s \in \mathbf{R}$ we denote by $W_p^s(\Omega)$ the L_p Sobolev space of order s, and by $W_{p,0}^s(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W_p^s(\Omega)$. In particular, if $s = -k$ with a positive integer k, the space $W_p^{-k}(\Omega)$ is the set of all functions u which are written as

$$(1.3)\qquad u = \sum_{|\alpha|\le k} \partial^\alpha u_\alpha, \quad u_\alpha \in L_p(\Omega),$$

and the norm $\|u\|_{W_p^{-k}(\Omega)}$ is equivalent to

$$\inf \sum_{|\alpha|\le k} \|u_\alpha\|_{L_p(\Omega)},$$

where the infimum is taken over all $\{u_\alpha\}_{|\alpha|\le k}$ satisfying (1.3).

Theorem 1. *Let $1 \le p < r < q \le \infty$, $p^{-1} - r^{-1} \le m/n$ and $r^{-1} - q^{-1} \le m/n$. In addition, let $p^{-1} - r^{-1} < m/n$ if $p = 1$, and let $r^{-1} - q^{-1} < m/n$ if $q = \infty$. Assume that T is a bounded linear operator from $W_r^{-m}(\Omega)$ to $W_r^m(\Omega)$. Then the following statements hold with*

$$\theta = (n/m)(p^{-1} - r^{-1}), \quad \eta = (n/m)(r^{-1} - q^{-1}).$$

(i) *T is a bounded operator from $L_p(\Omega)$ to $L_q(\Omega)$ and*

$$\|T\|_{L_p(\Omega)\to L_q(\Omega)} \le C\|T\|^{(1-\theta)(1-\eta)}_{L_r(\Omega)\to L_r(\Omega)}\|T\|^{\theta(1-\eta)}_{W_r^{-m}(\Omega)\to L_r(\Omega)} \times \|T\|^{(1-\theta)\eta}_{L_r(\Omega)\to W_r^m(\Omega)}\|T\|^{\theta\eta}_{W_r^{-m}(\Omega)\to W_r^m(\Omega)}$$

with $C = C(n, m, p, q, r, \Omega)$.

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(ii) *T is a bounded operator from $L_p(\Omega)$ to $W_r^m(\Omega)$ and*

$$\|T\|_{L_p(\Omega)\to W_r^j(\Omega)} \le C\|T\|^{1-\theta}_{L_r(\Omega)\to W_r^j(\Omega)}\|T\|^{\theta}_{W_r^{-m}\to W_r^j(\Omega)}$$

for $0 \le j \le m$ with $C = C(n, m, p, r, \Omega)$.

(iii) *T is a bounded operator from $W_r^{-m}(\Omega)$ to $L_q(\Omega)$ and*

$$\|T\|_{W_r^{-i}(\Omega)\to L_q(\Omega)} \le C\|T\|^{1-\eta}_{W_r^{-i}(\Omega)\to L_r(\Omega)}\|T\|^{\eta}_{W_r^{-i}\to W_r^m(\Omega)}$$

for $0 \le i \le m$ with $C = C(n, m, q, r, \Omega)$.

As is well known, a bounded linear operator T from $L_1(\Omega)$ to $L_\infty(\Omega)$ is written as

$$Tu(x) = \int_\Omega K(x,y)u(y)\,dy, \quad u \in L_1(\Omega)$$

with kernel $K(x,y) \in L_\infty(\Omega\times\Omega)$. If T satisfies a stronger condition, we can say more about its kernel. For a function $u(x)$ and $h \in \mathbf{R}^n$, we define the operator Δ_h by

$$\Delta_h u(x) = \begin{cases} u(x+h) - u(x) & (\text{if } x \in \Omega \text{ and } x+h \in \Omega),\\ 0 & (\text{otherwise}).\end{cases}$$

For a function $K(x,y)$ we write $\Delta_h^{(1)}$ (resp. $\Delta_h^{(2)}$) for Δ_h that operates $K(x,y)$ with respect to x (resp. y). Let $\mathbf{N}$ be the set of positive integers, and let $\mathbf{N}_0 = \mathbf{N}\cup\{0\}$.

Theorem 2. *Let $1 < p < \infty$, $m - n/p > 0$ and $p^{-1} + (p')^{-1} = 1$. Let $k \in \mathbf{N}_0$ and $0 < \tau < 1$ satisfy $m - n/p \ge k + \tau$. Assume that T is a bounded linear operator from $W_{p'}^{-m}(\Omega)$ to $W_p^m(\Omega)$. Then T is a bounded linear operator from $L_1(\Omega)$ to $L_\infty(\Omega)$, and the kernel $K(x,y)$ of T is in $C^k(\Omega\times\Omega)$. More precisely, for $|\alpha| \le k$ and $|\beta| \le k$ the derivatives $\partial_x^\alpha\partial_y^\beta K(x,y)$ are continuous and satisfy*

$$(1.4)\qquad |\partial_x^\alpha\partial_y^\beta K(x,y)| \le C\|T\|^{(1-\theta)(1-\eta)}_{L_{p'}(\Omega)\to L_p(\Omega)}\|T\|^{\theta(1-\eta)}_{L_{p'}(\Omega)\to W_p^m(\Omega)} \times \|T\|^{(1-\theta)\eta}_{W_{p'}^{-m}(\Omega)\to L_p(\Omega)}\|T\|^{\theta\eta}_{W_{p'}^{-m}(\Omega)\to W_p^m(\Omega)}$$

for $x, y \in \Omega$ with

$$(1.5)\qquad \theta = \frac{|\alpha| + np^{-1}}{m}, \quad \eta = \frac{|\beta| + np^{-1}}{m}$$

and $C = C(n, m, p, \Omega)$. Furthermore, the derivatives $\partial_x^\alpha\partial_y^\beta K(x,y)$ are Hölder continuous of order τ and satisfy

$$(1.6)\qquad |(\Delta_h^{(1)})^a(\Delta_h^{(2)})^b\partial_x^\alpha\partial_y^\beta K(x,y)| \le C|h|^\tau\|T\|^{(1-\theta)(1-\eta)}_{L_{p'}(\Omega)\to L_p(\Omega)}\|T\|^{\theta(1-\eta)}_{L_{p'}(\Omega)\to W_p^m(\Omega)} \times \|T\|^{(1-\theta)\eta}_{W_{p'}^{-m}(\Omega)\to L_p(\Omega)}\|T\|^{\theta\eta}_{W_{p'}^{-m}(\Omega)\to W_p^m(\Omega)}$$

for $x, y \in \Omega$, $h \in \mathbf{R}^n$ and $(a,b) = (1,0), (0,1)$ with

$$\theta = \frac{|\alpha| + a\tau + np^{-1}}{m}, \quad \eta = \frac{|\beta| + b\tau + np^{-1}}{m}$$

and $C = C(n, m, p, \tau, \Omega)$. Here $(\Delta_h^{(1)})^0$ and $(\Delta_h^{(2)})^0$ should be interpreted as the identity.

Remark 3. The estimate (1.4) with $\alpha = \beta = 0$ is considered to be a generalization of the kernel theorem [1, Lemma 3.2] for $p = 2$ to the case $p \ne 2$.

2. Proofs. For the proofs of Theorem 1 and Theorem 2 we use the Sobolev embedding theorem which guarantees the inclusions such as $W_p^m(\Omega) \subset L_q(\Omega)$, $L_p(\Omega) \subset W_q^{-m}(\Omega)$ and the inequalities for $\|u\|_{L_q(\Omega)}$, $\|u\|_{W_q^{-m}(\Omega)}$ if p and q satisfy suitable conditions. We need to formulate the embedding $L_p(\Omega) \subset W_q^{-m}(\Omega)$ more precisely than usual.

Lemma 4. *Let $1 \le p < q \le \infty$ and $m \ge n(p^{-1} - q^{-1})$. In addition, let $m > n(p^{-1} - q^{-1})$ if $p = 1$ or $q = \infty$.*

Let $u \in L_p(\Omega)$. Then for any $\lambda > 0$ there exist $v_\alpha \in L_q(\Omega)$ with $|\alpha| = m$ and $w \in L_q(\Omega)$ such that u is written as

$$(2.1)\qquad u = \sum_{|\alpha|=m}\partial^\alpha v_\alpha + w$$

and that

$$(2.2)\qquad \|v_\alpha\|_{L_q(\Omega)} \le C\lambda^{m-n(p^{-1}-q^{-1})}\|u\|_{L_p(\Omega)},$$

$$(2.3)\qquad \|w\|_{L_q(\Omega)} \le C\lambda^{-n(p^{-1}-q^{-1})}\|u\|_{L_p(\Omega)}$$

with $C = C(n, m, p, q, \Omega)$.

Proof. We may assume that $\Omega = \mathbf{R}^n$, since the case $\Omega \ne \mathbf{R}^n$ can be reduced to the case $\Omega = \mathbf{R}^n$ by extending $u \in L_p(\Omega)$ by zero to $\mathbf{R}^n$.

First we assume that u belongs to the Schwartz space $\mathcal{S}(\mathbf{R}^n)$. It is convenient to use Muramatu's integral formula [5], which expresses a function by its regularization. Let us briefly review it. Choose a function $\rho \in C_0^\infty(\mathbf{R}^n)$ satisfying $\int_{\mathbf{R}^n}\rho(x)\,dx = 1$ and $\operatorname{supp}\rho \subset \{x \in \mathbf{R}^n : |x| < 1\}$, and set

$$\varphi(x) = \sum_{|\alpha|<m}\frac{1}{\alpha!}\partial_x^\alpha\{x^\alpha\rho(x)\},$$

$$M(x) = \sum_{|\alpha|=m}M_\alpha^{(\alpha)}(x), \quad M_\alpha(x) = \frac{m}{\alpha!}x^\alpha\rho(x).$$

Here and in what follows we sometimes write $f^{(\alpha)}$ for the derivative $\partial^\alpha f$ of a function $f(x)$. For $t > 0$ and a function $f(x)$ we set $f_t(x) = t^{-n} f(t^{-1}x)$. Using the relations $\partial_t\{\varphi_t(x)\} = -t^{-1}M_t(x)$ and $\lim_{t\to+0} \varphi_t * u(x) = u(x)$, we have

$$(2.4) \quad u(x) = \int_0^\lambda M_t * u(x)\, \frac{dt}{t} + \varphi_\lambda * u(x), \quad \lambda > 0$$

for $u \in \mathcal{S}(\mathbf{R}^n)$, where the integral is an improper integral, namely, it is the limit of the Riemann integral $\int_\epsilon^\lambda M_t * u(x)\, t^{-1}dt$ as $\epsilon \to +0$. In view of $(M_\alpha^{(\alpha)})_t(x) = t^m \partial_x^\alpha (M_\alpha)_t(x)$ we see that (2.1) holds with

$$v_\alpha(x) = \int_0^\lambda (M_\alpha)_t * u(x) t^{m-1}\, dt,$$
$$w(x) = \varphi_\lambda * u(x).$$

Define $r > 1$ by $p^{-1} + r^{-1} = 1 + q^{-1}$. Then the Young inequality $\|w\|_{L_q} \le \|\varphi_\lambda\|_{L_r} \|u\|_{L_p}$ and $\|\varphi_\lambda\|_{L_r} = \lambda^{-n(p^{-1}-q^{-1})}\|\varphi\|_{L_r}$ give (2.3). If $m > n(p^{-1} - q^{-1}) > 0$, a similar calculation shows

$$\|v_\alpha\|_{L_q} \le \|M_\alpha\|_{L_r} \|u\|_{L_p} \int_0^\lambda t^{m-n(p^{-1}-q^{-1})-1}\, dt,$$

from which (2.2) follows.

If $m = n(p^{-1} - q^{-1})$, which implies $1 < p < q < \infty$ by assumption and therefore $0 < m < n$, the change of variables $|x - y|/t = s$ gives

$$\begin{aligned} |v_\alpha(x)| &\le \int_{\mathbf{R}^n} dy \int_0^\infty \left| M_\alpha\left(\frac{s(x-y)}{|x-y|}\right)\right| s^{n-m-1} \\ &\qquad \times |x-y|^{m-n} |u(y)|\, ds \\ &\le C \int_{\mathbf{R}^n} |x-y|^{m-n} |u(y)|\, dy. \end{aligned}$$

Hence the Hardy-Littlewood-Sobolev inequality yields (2.2).

Next, we consider the general case $u \in L_p(\mathbf{R}^n)$. We write T_α and S for the maps $u \mapsto v_\alpha$ and $u \mapsto w$, respectively, in the proof for the Schwartz function. From the result for the Schwartz function it follows that T_α and S, defined on $\mathcal{S}(\mathbf{R}^n)$, extend to bounded linear operators from $L_p(\mathbf{R}^n)$ to $L_q(\mathbf{R}^n)$. We choose a sequence of functions $(u_j)_{j\in\mathbf{N}}$ in $\mathcal{S}(\mathbf{R}^n)$ that converges to u in $L_p(\mathbf{R}^n)$. Then we have

$$(2.5) \qquad u_j = \sum_{|\alpha|=m} \partial^\alpha T_\alpha u_j + S u_j.$$

Since $T_\alpha u_j \to T_\alpha u$ and $Su_j \to Su$ in $L_q(\mathbf{R}^n)$ as $j \to \infty$, the right-hand side converges in $W_q^{-m}(\mathbf{R}^n)$. Hence (2.1) holds with $v_\alpha = T_\alpha u$ and $w = Su$. The inequalities (2.2) and (2.3) follow from the corresponding inequalities for the Schwartz function. □

Proof of Theorem 1. In any case, the boundedness of T follows by the Sobolev embedding theorem: $L_p(\Omega) \subset W_r^{-m}(\Omega)$ and $W_r^m(\Omega) \subset L_q(\Omega)$. So, it remains to evaluate the operator norms.

Let $u \in L_p(\Omega)$. By Lemma 4 there exist $v_\alpha \in L_r(\Omega)$ and $w \in L_r(\Omega)$ satisfying (2.1) and the inequalities similar to (2.2), (2.3). Then we have

$$Tu = \sum_{|\alpha|=m} T\partial^\alpha v_\alpha + Tw,$$

which gives

$$\begin{aligned} \|Tu\|_{W_r^j} &\le C\|T\|_{W_r^{-m}\to W_r^j} \lambda^{m-n(p^{-1}-r^{-1})} \|u\|_{L_p} \\ &\quad + C\|T\|_{L_r\to W_r^j} \lambda^{-n(p^{-1}-r^{-1})} \|u\|_{L_p} \end{aligned}$$

for $0 \le j \le m$. Minimizing the right-hand side if $m - n(p^{-1} - r^{-1}) > 0$, and letting $\lambda \to \infty$ if $m - n(p^{-1} - r^{-1}) = 0$, we get

$$(2.6) \qquad \|Tu\|_{W_r^j} \le C\|T\|_{L_r\to W_r^j}^{1-\theta} \|T\|_{W_r^{-m}\to W_r^j}^{\theta} \|u\|_{L_p}.$$

This inequality gives the estimate for (ii). The estimate for (iii) follows from the Sobolev inequality

$$(2.7) \qquad \|f\|_{L_q} \le C\|f\|_{L_r}^{1-\eta} \|f\|_{W_r^m}^{\eta}.$$

The estimate for (i) follows from (2.6) with $j = 0, m$ and (2.7). □

Lemma 5. *Let $1 < p < \infty$, $1/p + 1/p' = 1$ and $m - n/p > 0$. Let $\beta \in \mathbf{N}_0^n$ and $0 < \tau < 1$ satisfy $m - n/p \ge |\beta| + \tau$.*

Then for $u \in L_1(\mathbf{R}^n)$ and $\lambda > 0$ there exist $v_\gamma \in L_{p'}(\mathbf{R}^n)$ with $|\gamma| = m$ and $w \in L_{p'}(\mathbf{R}^n)$ such that $\partial^\beta u$ is written as

$$(2.8) \qquad \partial^\beta u = \sum_{|\gamma|=m} \partial^\gamma v_\gamma + w$$

and that

$$(2.9) \qquad \|v_\gamma\|_{L_{p'}(\mathbf{R}^n)} \le C\lambda^{m-|\beta|-n/p} \|u\|_{L_1(\mathbf{R}^n)},$$

$$(2.10) \qquad \|w\|_{L_{p'}(\mathbf{R}^n)} \le C\lambda^{-|\beta|-n/p} \|u\|_{L_1(\mathbf{R}^n)}$$

with $C = C(n, m, p)$. In addition, it holds that

$$(2.11) \qquad \|\Delta_h v_\gamma\|_{L_{p'}(\mathbf{R}^n)} \le C|h|^\tau \lambda^{m-|\beta|-n/p-\tau} \|u\|_{L_1(\mathbf{R}^n)},$$

$$(2.12) \qquad \|\Delta_h w\|_{L_{p'}(\mathbf{R}^n)} \le C|h|^\tau \lambda^{-|\beta|-n/p-\tau} \|u\|_{L_1(\mathbf{R}^n)}$$

for $h \in \mathbf{R}^n$ with $C = C(n, m, p, \tau)$.

Proof. We may assume $u \in \mathcal{S}(\mathbf{R}^n)$ by the same argument as in the proof of Lemma 4, since the maps $u \mapsto v_\gamma$ and $u \mapsto w$ which will be constructed below extend to bounded linear operators from $L_1(\mathbf{R}^n)$ to $L_{p'}(\mathbf{R}^n)$.

We apply Muramatu's formula (2.4) to $\partial^\beta u$, which belongs to $\mathcal{S}(\mathbf{R}^n)$, with in mind that

$$(M_\alpha^{(\alpha)})_t * \partial^\beta u(x) = t^{|\alpha|-|\beta|}\partial_x^\alpha\{(M_\alpha^{(\beta)})_t * u(x)\}.$$

Then (2.8) holds with

$$v_\gamma(x) = \int_0^\lambda (M_\gamma^{(\beta)})_t * u(x)\, t^{m-|\beta|-1}\, dt,$$
$$w(x) = \lambda^{-|\beta|}(\varphi^{(\beta)})_\lambda * u(x).$$

Then the same calculation as in the proof of Lemma 4 gives (2.9) and (2.10).

In order to derive (2.11) and (2.12) we note that

$$(2.13)\quad \|\Delta_h f_t\|_{L_{p'}} \le C(n)\|f\|_{W^1_{p'}} t^{-n/p}\min\{1,|h|/t\} \le C(n)\|f\|_{W^1_{p'}} t^{-n/p}(|h|/t)^\tau$$

for $f \in C_0^\infty(\mathbf{R}^n)$ and $f_t(x) = t^{-n}f(t^{-1}x)$ with $t>0$. The first inequality follows from $\|\Delta_h g\|_{L_{p'}} \le 2\|g\|_{L_{p'}}$ and $\Delta_h g(x) = \int_0^1 \nabla g(x+\theta h)\cdot h\, d\theta$ with $g = f_t$, and the second inequality is a consequence of $\min\{1,s\} \le s^\tau$ for $s>0$.

The second inequality in (2.13) yields (2.12). It also yields (2.11) if $m - n/p > |\beta| + \tau$. If $m - n/p = |\beta| + \tau$, we use the first inequality in (2.13) to get

$$\begin{aligned}\|\Delta_h v_\gamma\|_{L_{p'}} &\le \int_0^\lambda \|\Delta_h (M_\gamma^{(\beta)})_t\|_{L_{p'}}\|u\|_{L_1} t^{m-|\beta|-1}\, dt \\ &\le \int_0^\lambda C\min\{1,|h|/t\} t^{m-n/p-|\beta|-1}\, dt\, \|u\|_{L_1} \\ &\le C|h|^\tau \|u\|_{L_1}\int_0^\infty \min\{1,t^{-1}\}t^{\tau-1}\, dt,\end{aligned}$$

which gives (2.11). □

Proof of Theorem 2. First we assume $\Omega = \mathbf{R}^n$. Let $u \in L_1(\mathbf{R}^n)$ and $|\alpha| \le k$, $|\beta| \le k$. Taking into account that $W_p^m(\Omega) \subset C^{k+\tau}(\Omega)$ by the Sobolev embedding theorem, and using (2.8)–(2.10), we have

$$\begin{aligned}\|\partial^\alpha T\partial^\beta u\|_{L_\infty} &\le \sum_{|\gamma|=m}\|\partial^\alpha T\|_{W_{p'}^{-m}\to L_\infty}\|v_\gamma\|_{L_{p'}} \\ &\quad + \|\partial^\alpha T\|_{L_{p'}\to L_\infty}\|w\|_{L_{p'}} \\ &\le C\|\partial^\alpha T\|_{W_{p'}^{-m}\to L_\infty}\lambda^{m-|\beta|-n/p}\|u\|_{L_1} \\ &\quad + C\|\partial^\alpha T\|_{L_{p'}\to L_\infty}\lambda^{-|\beta|-n/p}\|u\|_{L_1}.\end{aligned}$$

Minimizing the last expression, we get

$$(2.14)\quad \|\partial^\alpha T\partial^\beta u\|_{L_\infty} \le C\|\partial^\alpha T\|^{1-\eta}_{L_{p'}\to L_\infty}\|\partial^\alpha T\|^\eta_{W_{p'}^{-m}\to L_\infty}\|u\|_{L_1}$$

with $\eta = (|\beta| + np^{-1})/m$. Hence $\partial^\alpha T\partial^\beta$ is a bounded operator from $L_1(\mathbf{R}^n)$ to $L_\infty(\mathbf{R}^n)$. We denote by $K^{\alpha\beta}(x,y)$ the kernel of $\partial^\alpha T\partial^\beta$, and simply write $K(x,y)$ for $K^{\alpha\beta}(x,y)$ with $\alpha=\beta=0$. It is easy to see that $\partial_x^\alpha\partial_y^\beta K(x,y) = (-1)^{|\beta|}K^{\alpha\beta}(x,y)$ in the distributional sense. The estimate for $\partial_x^\alpha\partial_y^\beta K(x,y)$ follows from (2.14) and the Sobolev inequality

$$(2.15)\quad \|\partial^\alpha f\|_{L_\infty} \le C\|f\|^{1-\theta}_{L_p}\|f\|^\theta_{W_p^m}$$

with $\theta = (|\alpha| + np^{-1})/m$.

In order to show the Hölder continuity of $K^{\alpha\beta}(x,y)$ we consider the operators $\Delta_h\partial^\alpha T\partial^\beta$ and $\partial^\alpha T\partial^\beta\Delta_h$. By the Lebesgue differentiation theorem we know that it is sufficient to obtain the inequalities similar to (1.6) for $\|\Delta_h^{(1)}K^{\alpha\beta}\|_{L_\infty}$ and $\|\Delta_h^{(2)}K^{\alpha\beta}\|_{L_\infty}$. Since the kernel of $\Delta_h\partial^\alpha T\partial^\beta$ is $\Delta_h^{(1)}K^{\alpha\beta}(x,y)$, (2.14) with ∂^α replaced by $\Delta_h\partial^\alpha$ and the Sobolev inequality

$$\|\Delta_h\partial^\alpha f\|_{L_\infty} \le C|h|^\tau\|f\|^{1-\theta}_{L_p}\|f\|^\theta_{W_p^m}$$

with $\theta = (|\alpha| + \tau + np^{-1})/m$ yield (1.6) for $(a,b) = (1,0)$.

Noting that $\partial^\beta\Delta_h = \Delta_h\partial^\beta$, and using (2.8), (2.11) and (2.12), we have

$$\begin{aligned}&\|\partial^\alpha T\partial^\beta\Delta_h u\|_{L_\infty} \\ &\le \sum_{|\gamma|=m}\|\partial^\alpha T\|_{W_{p'}^{-m}\to L_\infty}\|\Delta_h v_\gamma\|_{L_{p'}} \\ &\quad + \|\partial^\alpha T\|_{L_{p'}\to L_\infty}\|\Delta_h w\|_{L_{p'}} \\ &\le C|h|^\tau\lambda^{m-|\beta|-n/p-\tau}\|\partial^\alpha T\|_{W_{p'}^{-m}\to L_\infty}\|u\|_{L_1} \\ &\quad + C|h|^\tau\lambda^{-|\beta|-n/p-\tau}\|\partial^\alpha T\|_{L_{p'}\to L_\infty}\|u\|_{L_1}.\end{aligned}$$

Minimizing the last expression, we get

$$(2.16)\quad \|\partial^\alpha T\partial^\beta\Delta_h u\|_{L_\infty} \le C|h|^\tau\|\partial^\alpha T\|^{1-\eta}_{L_{p'}\to L_\infty}\|\partial^\alpha T\|^\eta_{W_{p'}^{-m}\to L_\infty}\|u\|_{L_1}$$

with $\eta = (|\beta| + \tau + np^{-1})/m$. Since the kernel of $\partial^\alpha T\partial^\beta\Delta_h$ is $\Delta_{-h}^{(2)}K^{\alpha\beta}(x,y)$, (2.15) and (2.16) yield (1.6) for $(a,b) = (0,1)$.

We see that $K(x,y) \in C^k(\mathbf{R}^n\times\mathbf{R}^n)$ from the continuity of $K^{\alpha\beta}(x,y)$ for $|\alpha| \le k$, $|\beta| \le k$.

Next, we consider the case $\Omega \ne \mathbf{R}^n$. Let $u \in L_1(\Omega)$. Let E be the universal extension operator for the Sobolev spaces on Ω to the corresponding spaces on $\mathbf{R}^n$, and let R be the restriction to Ω. We denote by $\tilde{K}(x,y)$ the kernel of the bounded operator $ETR\colon W_{p'}^{-m}(\mathbf{R}^n) \to W_p^m(\mathbf{R}^n)$. We define E_0u by $E_0u(x) = u(x)$ for $x \in \Omega$ and $E_0u(x) = 0$ for $x \in \Omega^c$. Since $Tu = R(ETR)E_0u$, the kernel of T is given by $\tilde{K}|_{\Omega\times\Omega}$. Hence the case $\Omega \ne \mathbf{R}^n$ reduces to the case $\Omega = \mathbf{R}^n$. □

3. Application. Before applying Theorem 1 and Theorem 2 to the elliptic operator A defined in (1.1), we precisely describe the assumptions on A. We assume that A satisfies the following conditions:

(i) The principal symbol

$$a(x,\xi) = \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\xi^{\alpha+\beta}$$

of A satisfies the strong ellipticity condition, i.e., there exists $\delta_A > 0$ such that

$$\operatorname{Re} a(x,\xi) \geq \delta_A |\xi|^{2m} \quad \text{for } x \in \Omega,\ \xi \in \mathbf{R}^n.$$

(ii) All the coefficients $a_{\alpha\beta}$ are in $L_\infty(\Omega)$, and the leading coefficients are uniformly continuous in Ω.

By assumption there exists $\omega_A \in [0,\pi/2)$ such that

$$|\arg a(x,\xi)| \leq \omega_A \quad \text{for } x \in \Omega,\ \xi \in \mathbf{R}^n.$$

For each $p \in (1,\infty)$ the operator A is regarded as a bounded linear operator

$$W^m_{p,0}(\Omega) \to W^{-m}_p(\Omega).$$

For $R > 0$ and $\omega \in (0,\pi/2)$ we set

$$\Lambda(R,\omega) = \{\lambda \in \mathbf{C} :\ |\lambda| \geq R,\ \omega \leq \arg\lambda \leq 2\pi - \omega\}.$$

Theorem 6. *Let $\omega \in (\omega_A, \pi/2)$ be given. Then there exist $R = R(n,m,\omega,A,\Omega)$ such that the inverse of the operator*

$$A - \lambda :\ \cup_{1<p<\infty} W^m_{p,0}(\Omega) \to \cup_{1<p<\infty} W^{-m}_p(\Omega)$$

exists for $\lambda \in \Lambda(R,\omega)$ and that for each $p \in (1,\infty)$ the inverse $(A-\lambda)^{-1}$ is a bounded operator from $W^{-m}_p(\Omega)$ to $W^m_{p,0}(\Omega)$ that satisfies

$$\|(A-\lambda)^{-1}\|_{W^{-i}_p(\Omega)\to W^j_p(\Omega)} \leq C|\lambda|^{-1+(i+j)/2m} \tag{3.1}$$

for $0 \leq i \leq m$, $0 \leq j \leq m$ with $C = C(n,m,p,\omega,A,\Omega)$.

Proof. See [2] for a uniform C^{m+1} domain and [3] for a uniform C^1 domain. □

In [2] we obtained Theorem 6 via the Gaussian estimates for heat kernels and the exponential decay estimates for resolvent kernels from its weak version which is the same as Theorem 6 except that the constant R may depend on p. In the process of obtaining Theorem 6 from its weak version we essentially proved and utilized

$$\|(A-\lambda)^{-1}\|_{L_p(\Omega)\to L_q(\Omega)} \leq C|\lambda|^{-1+(n/2m)(p^{-1}-q^{-1})} \tag{3.2}$$

for $1 < p < q < \infty$ and $p^{-1} - q^{-1} < m/n$, and

$$\|(A-\lambda)^{-N}\|_{L_1(\Omega)\to L_\infty(\Omega)} \leq C|\lambda|^{-N+n/2m} \tag{3.3}$$

for $N > 2 + n/m$. Since we used (3.1) only for $i = 0$ to derive (3.2) and (3.3), the conditions on p, q and N in (3.2) and (3.3) may be restrictive. Theorem 1 and Theorem 2 enable us to make a full use of (3.1) and relax the conditions on p, q and N, as shown below. The improvement for the conditions on p, q and N is of interest in itself, although it does not improve the statement of Theorem 6.

Corollary 7. *Given $\omega \in (\omega_A, \pi/2)$, let R be the constant in Theorem 6 and let $\lambda \in \Lambda(R,\omega)$.*

(i) *Let $N \in \mathbf{N}$, $1 \leq p < q \leq \infty$ and $p^{-1} - q^{-1} \leq 2mN/n$. In addition, let $p^{-1} - q^{-1} < 2mN/n$ if $p = 1$ or $q = \infty$. Then $(A-\lambda)^{-N}$ is a bounded operator from $L_p(\Omega)$ to $L_q(\Omega)$ and satisfies*

$$\|(A-\lambda)^{-N}\|_{L_p(\Omega)\to L_q(\Omega)} \leq C|\lambda|^{-N+(n/2m)(p^{-1}-q^{-1})}$$

with $C = C(n,m,p,q,\omega,N,A,\Omega)$.

(ii) *Let $N \in \mathbf{N}$, $2mN > n$, and take $k \in \mathbf{N}_0$ and $0 < \tau < 1$ so that $0 \leq k < m$ and $2mN \geq n + 2(k+\tau)$. Then $(A-\lambda)^{-N}$ is a bounded operator from $L_1(\Omega)$ to $L_\infty(\Omega)$ and its kernel $G^N_\lambda(x,y)$ is in $C^k(\Omega\times\Omega)$. More precisely, for $|\alpha| \leq k$ and $|\beta| \leq k$ the derivatives $\partial^\alpha_x\partial^\beta_y G^N_\lambda(x,y)$ are continuous and satisfy*

$$|\partial^\alpha_x\partial^\beta_y G^N_\lambda(x,y)| \leq C_1|\lambda|^{-N+(n+|\alpha|+|\beta|)/2m} \tag{3.4}$$

for $x,y \in \Omega$ with $C_1 = C_1(n,m,\omega,N,A,\Omega)$. Furthermore, the derivatives $\partial^\alpha_x\partial^\beta_y G^N_\lambda(x,y)$ are Hölder continuous of order τ and satisfy

$$|\Delta^{(1)}_h \partial^\alpha_x\partial^\beta_y G^N_\lambda(x,y)| + |\Delta^{(2)}_h \partial^\alpha_x\partial^\beta_y G^N_\lambda(x,y)| \leq C_2|h|^\tau|\lambda|^{-N+(n+|\alpha|+|\beta|+\tau)/2m} \tag{3.5}$$

for $x,y \in \Omega$ and $h \in \mathbf{R}^n$ with $C_2 = C_2(n,m,\omega,\tau,N,A,\Omega)$.

Remark 8. Corollary 7(i) is essentially the same as [4, Lemma 3.4] whose proof is also based on Theorem 6, but heavily relies on the exponential decay estimates for resolvent kernels.

Proof. (i) First let $N = 1$. Define r so that $r^{-1} = (p^{-1} + q^{-1})/2$, which implies $p^{-1} - r^{-1} \leq m/n$ and $r^{-1} - q^{-1} \leq m/n$. It also holds that $p^{-1} - r^{-1} < m/n$ and $r^{-1} - q^{-1} < m/n$ if $p = 1$ or $q = \infty$. Using Theorem 1(i) and Theorem 6, we see that $(A-\lambda)^{-1}$ is a bounded operator from $L_p(\Omega)$ to $L_q(\Omega)$ and get, with $\theta = (n/2m)(p^{-1} - q^{-1})$,

$$\begin{aligned}\|(A-\lambda)^{-1}\|_{L_p\to L_q} &\leq C|\lambda|^{-1}(|\lambda|^0)^{(1-\theta)^2}(|\lambda|^{1/2})^{\theta(1-\theta)} \\ &\quad \times (|\lambda|^{1/2})^{(1-\theta)\theta}(|\lambda|^1)^{\theta^2} \\ &\leq C|\lambda|^{-1+\theta}.\end{aligned}$$

The case $N \geq 2$ is treated by using the result for $N = 1$ repeatedly.

(ii) Choose p and a sequence $(p_l)_{l=0}^N$ so that $p = p_0 = p_1 = 2$ if $N = 1$, and so that

$$p/(p-1) = p_0 < p_1 < p_2 < \cdots < p_N = p < \infty,$$
$$m - n/p > k,$$
$$p_0^{-1} - p_1^{-1} \leq m/n, \quad p_{N-1}^{-1} - p_N^{-1} \leq m/n,$$
$$p_{l-1}^{-1} - p_l^{-1} \leq 2m/n \quad (2 \leq l \leq N-1)$$

if $N \geq 2$. Evaluating $\|(A-\lambda)^{-1}\|_{W_{p_0}^{-i} \to L_{p_1}}$, $\|(A-\lambda)^{-1}\|_{L_{p_{N-1}} \to W_{p_N}^j}$, and $\|(A-\lambda)^{-1}\|_{L_{p_{l-1}} \to L_{p_l}}$ with $2 \leq l \leq N-1$ by Theorem 1, we see that $(A-\lambda)^{-N}$ is a bounded operator from $W_{p'}^{-m}(\Omega)$ to $W_p^m(\Omega)$ with $p' = p/(p-1)$ and get

$$\begin{aligned}&\|(A-\lambda)^{-N}\|_{W_{p'}^{-i} \to W_p^j}\\ &\quad \leq C|\lambda|^{-N+(i+j)/2m+(n/2m)((p')^{-1}-p^{-1})}\end{aligned}$$

for $0 \leq i \leq m$, $0 \leq j \leq m$. By Theorem 2 we obtain, with θ and η defined in (1.5),

$$\begin{aligned}&|\partial_x^\alpha \partial_y^\beta G_\lambda^N(x,y)|\\ &\quad \leq C|\lambda|^{-N+(n/2m)((p')^{-1}-p^{-1})}(|\lambda|^0)^{(1-\theta)(1-\eta)}\\ &\qquad \times (|\lambda|^{1/2})^{\theta(1-\eta)}(|\lambda|^{1/2})^{(1-\theta)\eta}(|\lambda|^1)^{\theta\eta}\\ &\quad \leq C|\lambda|^{-N+(n/2m)(1-2/p)+(\theta+\eta)/2},\end{aligned}$$

which yields (3.4).

Similarly we obtain (3.5) if we replace $m - n/p > k$ by $m - n/p \geq k + \tau$ in the definition of the sequence $(p_l)_{l=0}^N$. □

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