Some examples causing energy growth for solutions to wave equations

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Abstract: In this paper we study energy growth for solutions to wave equations. We prove that there exist compact in space perturbations of the wave equation $\partial_t^2 u - \triangle u = 0$ such that the energy of solution grows at the rate $\exp((1+t)^\alpha)$ for any $\alpha \geq 0$.

Key words: Wave equation; energy growth; compact in space perturbation.

1. Introduction. We are interested in energy growth for solutions to wave equations. There are many results about lower bounds of energy. For instance Reissig-Yagdjian [5] showed that there is an exponentially growing solution to

$$\partial_t^2 u - a(t)^2 \triangle u = 0,$$

where $a(t)$ is positive, smooth, periodic and nonconstant.

On the other hand, for solutions to wave equations in divergence form

$$\partial_t^2 u - \sum_{i,j=1}^{n} \partial_{x_i}(a_{i,j}(x)\partial_{x_j} u) = 0,$$

the energy is preserved.

In this paper we consider compact in space perturbation cases, that is

$$\partial_t^2 u - \sum_{i,j=1}^{n} \partial_{x_i}(a_{i,j}(t,x)\partial_{x_j} u) = 0,$$

where $a_{i,j}(t,x)$ is constant outside a compact set in $\mathbf{R}_x^n$.

Colombini-Rauch [1] studied an example of compact in space perturbation that would give exponentially growing solutions. But their proof was not complete. Doi-Nishitani-Ueda [2] completed thier proof and extended this result to examples that give $\exp((1+t)^\alpha)$ growth of energy for any $0 \leq \alpha \leq 1$.

Here we shall further extend these results and we get examples that give $\exp((1+t)^\alpha)$ growth of energy for any $0 \leq \alpha$.

2. Main result. We consider the wave equation

$$\partial_t^2 u - \sum_{i,j=1}^{n} \partial_{x_i}(a_{i,j}(t,x)\partial_{x_j} u) = 0 \tag{2.1}$$

with Cauchy data

$$u(0,x) = f(x), \quad \partial_t u(0,x) = g(x),$$

where $t \in [0,\infty), x \in \mathbf{R}^n$ and u denotes a complex-valued unknown function. We assume that $a_{i,j}(t,x)$ are smooth, real-valued, $a_{i,j} = a_{j,i}$, and there exist a constant $A > 0$ and a smooth nonnegative function $\delta(t)$ such that for any $(t,x,\xi) \in [0,\infty) \times \mathbf{R}^n \times \mathbf{R}^n$ we have

$$\begin{aligned} A^{-2}|\xi|^2 &\leq \sum_{i,j=1}^{n} a_{i,j}(t,x)\xi_i\xi_j \\ &\leq A^2(1+\delta(t))^2|\xi|^2. \end{aligned} \tag{2.2}$$

Moreover, we assume that for any multiindex $\alpha \in \mathbf{Z}_{\geq 0}^n$ the inequality

$$|\partial_x^\alpha a_{i,j}(t,x)| \leq C_\alpha(1+\delta(t))^2 \tag{2.3}$$

holds with some constant $C_\alpha > 0$. We put

$$a(t,x,\xi) := \sum_{i,j=1}^{n} a_{i,j}(t,x)\xi_i\xi_j$$

and

$$E(u,t) := \int_{\mathbf{R}^n} |\partial_t u|^2 + \sum_{i,j=1}^{n} a_{i,j}\partial_{x_i} u \overline{\partial_{x_j} u} dx.$$

We call $E(u,t)$ the total energy of u. If $f, g \in C_0^\infty(\mathbf{R}^n)$, then the energy identity holds:

$$\begin{aligned} E(u,t) &= E(u,0) \\ &+ \int_0^t \int \sum_{i,j=1}^{n} \partial_t a_{i,j}(s,x)\partial_{x_i} u \overline{\partial_{x_j} u} dx ds. \end{aligned} \tag{2.4}$$

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Denote by $\mathcal{H}$ the Hilbert space that is the completion of $C_0^\infty(\mathbf{R}^n)$ with respect to the norm

$$\|u\|_{\mathcal{H}}^2 := \int_{\mathbf{R}^n} \sum_{i=1}^n |\partial_{x_i} u|^2 dx = \int_{\mathbf{R}^n} |\nabla u|^2 dx.$$

Let $\mathcal{R}(t,0)$ be the solution operator defined by

$$\begin{array}{ccc} C_0^\infty(\mathbf{R}^n) \times C_0^\infty(\mathbf{R}^n) & \to & C_0^\infty(\mathbf{R}^n) \times C_0^\infty(\mathbf{R}^n) \\ (u(0,\cdot), \partial_t u(0,\cdot)) & \mapsto & (u(t,\cdot), \partial_t u(t,\cdot)) \end{array},$$

which can be extended uniquely to bounded operator on $\mathcal{H} \times L^2(\mathbf{R}^n)$. A bicharacteristic of a function $H(t,x,\xi)$ is a solution to the canonical equation

$$(2.5) \qquad \begin{cases} \dfrac{dX}{dt}(t) = \nabla_\xi H(t, X(t), \Xi(t)), \\ \dfrac{d\Xi}{dt}(t) = -\nabla_x H(t, X(t), \Xi(t)). \end{cases}$$

Following Colombini-Rauch [1] and Doi-Nishitani-Ueda [2], we use the following lower estimate of energy in terms of a null bicharacteristic:

Lemma 2.1. *Assume that there is a bicharacteristic $(X(t), \Xi(t))$ of $\sqrt{a}$ or $-\sqrt{a}$ such that*

$$(2.6) \qquad |\Xi(t)| > c^*$$

for $t \geq 0$ with some constant $c^* > 0$. Then there exists a family of Cauchy data $(f_j, g_j) \in \mathcal{S}(\mathbf{R}^n) \times \mathcal{S}(\mathbf{R}^n)(j \in \mathbf{N})$ such that for corresponding solutions u_j to (2.1) *we have*

$$\limsup_{j\to\infty} \frac{E(u_j,t)}{E(u_j,0)} \geq \frac{1}{4} G(t)$$

for all $t \geq 0$, where

$$G(t) = \exp\biggl(\int_0^t \frac{\partial_t a}{2a}(s, X(s), \Xi(s)) ds\biggr).$$

Here $\mathcal{S}(\mathbf{R}^n)$ denotes the Schwartz space on $\mathbf{R}^n$.

We can prove this lemma in the same way as in Nishiyama [4], where he treated the case $a_{i,j}(t,x) = a_{i,j}(x)$ with a damping term.

From this lemma, we have the following estimate of the operator norm of $\mathcal{R}(t,0)$:

Corollary 2.2. *We have*

$$\|\mathcal{R}(t,0)\|_{\mathcal{L}(\mathcal{H}\times L^2)} \geq \frac{1}{2} A^{-2}(1+\delta(t))^{-1}\sqrt{G(t)}$$
$$\geq \frac{1}{2} A^{-3}(1+\delta(0))^{-1/2}(1+\delta(t))^{-1}\sqrt{\frac{|\Xi(t)|}{|\Xi(0)|}}$$

for all $t \geq 0$.

Applying Corollary 2.2, we can construct examples which cause $\exp(\int_0^t \delta(s)ds)$ growth of energy for any given $\delta(t)$. Our construction works in all dimensions $n \geq 2$ though we present only the case $n = 2$ for simplicity. Consider the wave equation

$$(2.7) \qquad \partial_t^2 u - \sum_{i=1}^2 \partial_{x_i}(\tilde{a}(t,x)\partial_{x_i} u) = 0,$$

where $(t,x) \in [0,\infty) \times \mathbf{R}^2$. That is, $a_{12} = a_{21} = 0$ and $a_{11} = a_{22} = \tilde{a}$ in (2.1). To apply Corollary 2.2, we require the following conditions that correspond to (2.2), (2.3) and that $\tilde{a}$ is compact in space perturbation:

$$(2.8) \qquad \begin{aligned} &\tilde{a}(t,x) \equiv 1 \quad \text{for} \quad |x| \geq 2, \\ &A^{-2} \leq \tilde{a}(t,x) \leq A^2(1+\delta(t))^2, \\ &|\partial_x^\alpha \tilde{a}(t,x)| \leq C_\alpha(1+\delta(t))^2. \end{aligned}$$

Theorem 2.3. *For any smooth nonnegative function $\delta(t)$ on $[0,\infty)$ there exists $\tilde{a}(t,x)$ satisfying* (2.8) *such that for the associate solution operator $\mathcal{R}(t,0)$ to* (2.7) *we have*

$$(2.9) \qquad \|\mathcal{R}(t,0)\|_{\mathcal{L}(\mathcal{H}\times L^2)} \geq \frac{1}{2} A^{-3}(1+\delta(0))^{-1/2} \times (1+\delta(t))^{-1} \times \exp\biggl(\int_0^t \delta(s)ds\biggr).$$

Moreover, if $\delta(t)$ satisfies

$$(2.10) \qquad \inf_{t\in[0,\infty)} \delta(t) > 0, \quad \delta' = o(\delta^2), \quad \delta'' = o(\delta^3)$$

as $t \to \infty$, then for any $\varepsilon > 0$ there exists $\tilde{a}(t,x)$ satisfying (2.8) *such that for the associate solution operator $\mathcal{R}(t,0)$ to* (2.7) *we have*

$$(2.11) \qquad \begin{aligned} &\frac{1}{2} A^{-3}\delta(0)^{-1/2}\delta(t)^{-1} \exp\biggl(\int_0^t \delta(s)ds\biggr) \\ &\leq \|\mathcal{R}(t,0)\|_{\mathcal{L}(\mathcal{H}\times L^2)} \\ &\leq C_0 \exp\biggl((2+\varepsilon)\int_0^t \delta(s)ds\biggr), \end{aligned}$$

$$(2.12) \qquad \begin{aligned} &C_1 \delta(t)^{-1} \exp\biggl(\int_0^t \delta(s)ds\biggr) \\ &\leq \|\mathcal{R}(t,0)\|_{\mathcal{L}(H^1\times L^2;\mathcal{H}\times L^2)} \\ &\leq \|\mathcal{R}(t,0)\|_{\mathcal{L}(H^1\times L^2)} \\ &\leq C_2 \exp\biggl((1+\varepsilon)\int_0^t \delta(s)ds\biggr), \end{aligned}$$

where H^1 denotes the usual Sobolev space.

We note that when $\delta(t)$ is bounded, estimates (2.11), (2.12) are proved in [2].

3. Proof of Lemma 2.1. This proof is almost similar to the proof in Nishiyama [4]. We first describe h-pseudodifferential operators, microlocal defect measures and a diagonalization of the equation (2.1). Let h be a small positive parameter, $S^m = S((1+|\xi|^2)^{m/2}, |dx|^2 + (1+|\xi|^2)^{-1}|d\xi|^2)$ and $a \in S^m$. We define

$$a_h^w u(x) = \frac{1}{(2\pi h)^n} \iint e^{i(x-y)\xi/h} a\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi$$

for $u \in \mathcal{S}(\mathbf{R}^n)$. The operator a_h^w is called the h-pseudodifferential operator of symbol a. We put

$$OPS^m = \{a_h^w; a \in S^m\},$$
$$h^\infty S^{-\infty} = \bigcap_{r\in\mathbf{R}} \bigcap_{l\in\mathbf{R}} h^r S^l,$$
$$h^\infty OPS^{-\infty} = \{a_h^w; a \in h^\infty S^{-\infty}\}.$$

If $\{u_h\}$ is a bounded family in $L^2(\mathbf{R}^n)$ then there exist a subsequence $\{h_j\}_{j\in\mathbf{N}}$ tending to 0 and a positive Radon measure μ on $\mathbf{R}^{2n}$ such that

$$\lim_{j\to\infty} (a_{h_j}^w u_{h_j}, u_{h_j}) = \int_{\mathbf{R}^{2n}} a(x,\xi) d\mu$$

for any $a \in C_0^\infty(\mathbf{R}^{2n})$. We call μ the microlocal defect measure associated with $\{u_{h_j}\}$.

Next, we explain a diagonalization of (2.1). Let $\chi(\xi) \in C_0^\infty(\mathbf{R}^n)$ satisfy $0 \le \chi \le 1$, $\mathrm{supp}\chi \subset \{|\xi| < c^*\}$ and $\chi \equiv 1$ near 0. Multiplying (2.1) by h and adding $(1/h)\chi_h^w u$ we have

$$h\partial_t^2 u + \frac{1}{h}\left(a(t,x,\xi) + \frac{h^2}{4}\sum_{i,j} \partial_i\partial_j a_{i,j} + \chi(\xi)\right)_h^w u = \frac{1}{h}\chi_h^w u.$$

By ellipticity of $a + (h^2/4)\sum \partial_i\partial_j a_{i,j} + \chi$, one can find $\lambda \in S^1$ satisfying

$$\left(a(t,x,\xi) + \frac{h^2}{4}\sum_{i,j} \partial_i\partial_j a_{i,j}(t,x) + \chi(\xi)\right)_h^w \equiv \lambda_h^w \circ \lambda_h^w \mod h^\infty OPS^{-\infty}.$$

We put

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \partial_t + \frac{i}{h}\lambda_h^w \\ \partial_t - \frac{i}{h}\lambda_h^w \end{pmatrix} u,$$

then V is a solution to

$$h\partial_t V = \begin{pmatrix} i\lambda_h^w & 0 \\ 0 & -i\lambda_h^w \end{pmatrix} V + \frac{h}{2}\left(\frac{\partial_t \lambda}{\lambda}\right)_h^w \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} V - \frac{h^2}{4i}\left(\frac{1}{\lambda^2}\{\partial_t\lambda, \lambda\}\right)_h^w \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} V + R\begin{pmatrix} u \\ u \end{pmatrix},$$

where R denotes the remainder and $\{\cdot,\cdot\}$ the Poisson bracket:

$$\{a, b\} = \partial_\xi a \partial_x b - \partial_x a \partial_\xi b.$$

In order to diagonalize the principal term of the equation above. We introduce

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{ih}{4}\left(\frac{\partial_t\lambda}{\lambda^2} + \frac{b}{\lambda}\right)_h^w \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$\Lambda = \begin{pmatrix} i\lambda_h^w & 0 \\ 0 & -i\lambda_h^w \end{pmatrix} + \frac{h}{2}\left(\frac{\partial_t\lambda}{\lambda} - b\right)_h^w \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then QV satisfies

$$h\partial_t(QV) = \Lambda QV + \tilde{R}V + QR\begin{pmatrix} u \\ u \end{pmatrix},$$

where $\tilde{R} \in h^2 OPS^{-1}$. Let $\tilde{\chi}(\xi) \in C_0^\infty(\mathbf{R}^n)$ satisfy $0 \le \tilde{\chi} \le 1$, $\mathrm{supp}\tilde{\chi} \subset \{|\xi| < c^*\}$ and $\tilde{\chi} \equiv 1$ near 0. We put

$$W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = (1-\tilde{\chi})_h^w QV$$

then we have

$$h\partial_t W \equiv \Lambda W + [(1-\tilde{\chi})_h^w, \Lambda]W \mod h^2 OPS^{-1}V,\ h^2 OPS^{-1}u,$$

where $[\cdot,\cdot]$ denotes the commutator. If we take a family $\{W_h\}$ that satisfies the equation above, then the microlocal defect measure ν of $\{W_h\}$ satisfies certain corresponding equation. More precisely, we have

Lemma 3.1. *Assume that $(f_h, g_h) \in \mathcal{S}(\mathbf{R}^n) \times \mathcal{S}(\mathbf{R}^n)$, $\|f_h\|_{L^2(\mathbf{R}^n)} = O(h^{-\infty})$ and $\{u_h\}$ are the corresponding solutions to (2.1). We define $\{v_{1,h}\}$, $\{v_{2,h}\}$, $\{w_{1,h}\}$, $\{w_{2,h}\}$ from $\{u_h\}$ as above. If $\sup_{h\in(0,1]} E(u_h, 0) < +\infty$ then one can find a subsequence $\{h_j\}_{j\in\mathbf{N}}$ tending to 0 and microlocal defect measures $\nu_k(t)$ on $\mathbf{R}^n \times \mathbf{R}^n_{|\xi|>c^*}$ associated with $\{w_{k,h_j}(t)\} (k = 1, 2)$ such that*

$$\begin{cases} \dfrac{d}{dt}\nu_1 = \{\sqrt{a}, \nu_1\} + \dfrac{\partial_t a}{2a}\nu_1, \\ \dfrac{d}{dt}\nu_2 = -\{\sqrt{a}, \nu_2\} + \dfrac{\partial_t a}{2a}\nu_2 \end{cases}$$

on $\mathbf{R}^n \times \mathbf{R}^n_{|\xi|>c^*}$ in the sense of distribution. Here $\mathbf{R}^n_{|\xi|>c^*} = \{\xi \in \mathbf{R}^n; |\xi| > c^*\}$.

We can prove this lemma by differentiating the form of microlocal defect measure. We omit the detail, which can be found in Nishiyama [4].

Now we prove Lemma 2.1:

Proof of Lemma 2.1. We can assume that $(X(t), \Xi(t))$ is a bicharacteristic of $-\sqrt{a}$ without loss of generality. We put $(x_0,\xi_0)=(X(0),\Xi(0))$. Let $\varphi\in C_0^\infty(\mathbf{R}^n)$ satisfy $\|\varphi\|_{L^2(\mathbf{R}^n)}=1$ and we define

$$\Phi_h(x)=h^{-4/n}\varphi\left(\frac{x-x_0}{h^{1/2}}\right)e^{ix\cdot\xi_0/h}.$$

We choose Cauchy data

$$(f_h,g_h)=\left(\frac{h}{i}(\lambda_h^w)^{-1}\frac{1}{2}\Phi_h,\frac{1}{2}\Phi_h\right),$$

where $(\lambda_h^w)^{-1}$ is a parametrix of λ_h^w. We note that $(v_{1,h}(0),v_{2,h}(0))\equiv(\Phi_h,0)\mod h^\infty S^{-\infty}$. Using the sharp Gårding inequality, we calculate

(3.1)

$$\begin{aligned}
&E(u_h,0)\\
&\quad=\frac{1}{4}\sum_{i,j}(a_{i,j}h\partial_j(\lambda_h^w)^{-1}\Phi_h,h\partial_i(\lambda_h^w)^{-1}\Phi_h)_{L^2}+\frac{1}{4}\\
&\quad=-\frac{1}{4}\sum_{i,j}((\lambda_h^w)^{-1}h\partial_i a_{i,j}h\partial_j(\lambda_h^w)^{-1}\Phi_h,\Phi_h)_{L^2}+\frac{1}{4}\\
&\quad=\frac{1}{2}-(\chi_h^w(\lambda_h^w)^{-1}\Phi_h,(\lambda_h^w)^{-1}\Phi_h)_{L^2}+O(h^\infty)\\
&\quad\le\frac{1}{2}+O(h).
\end{aligned}$$

It is known that $\{\Phi_h\}$ has the defect measure $\delta_{(x_0,\xi_0)}$ (see Evans-Zworski [3]) and it is easy to see that $(1-\tilde\chi)_h^w v_{1,h}(0)$ and $w_{1,h}(0)$ have same defect measure $\delta_{(x_0,\xi_0)}$. From this and Lemma 3.1, we can take a subsequence $\{h_j\}_{j\in\mathbf{N}}$ tending to 0 and microlocal defect measure $\nu_1(t)$ on $\mathbf{R}^n\times\mathbf{R}^n_{|\xi|>c^*}$ associated with $\{w_{1,h_j}(t)\}$ satisfying

$$\begin{cases}\dfrac{d}{dt}\nu_1=\{\sqrt{a},\nu_1\}+\dfrac{\partial_t a}{2a}\nu_1,\\ \nu_1(0)=\delta_{(x_0,\xi_0)}.\end{cases}$$

Solving this equation we have

$$\nu_1(t)=G(t)\delta_{(X(t),\Xi(t))}.$$

Since

$$\begin{aligned}
G(t)&=\int_{\mathbf{R}^n\times\mathbf{R}^n_{|\xi|>c^*}}G(t)\delta_{(X(t),\Xi(t))}dxd\xi\\
&=\int_{\mathbf{R}^n\times\mathbf{R}^n_{|\xi|>c^*}}d\nu_1(t)\le\lim_{j\to\infty}\|w_{1,h_j}(t)\|^2_{L^2(\mathbf{R}^n)}
\end{aligned}$$

and

$$\begin{aligned}
\|w_{1,h_j}(t)\|^2_{L^2(\mathbf{R}^n)}&\le2\|w_{1,h_j}(t)-(1-\tilde\chi)_h^w v_{1,h_j}(t)\|^2_{L^2(\mathbf{R}^n)}\\
&\quad+2\|(1-\tilde\chi)_h^w v_{1,h_j}(t)\|^2_{L^2(\mathbf{R}^n)}\\
&\le4E(u_{h_j},t)+O(h)
\end{aligned}$$

(see Nishiyama [4] Lemma 3.1), we obtain

$$4\limsup_{j\to\infty}E(u_{h_j},t)\ge G(t).$$

On the other hand, from (3.1) it follows that $E(u_h,0)\le1$ for small h. Thus we have

$$4\frac{E(u_h,t)}{E(u_h,0)}\ge4E(u_h,t)$$

for small h. Consequently we obtain

$$4\limsup_{j\to\infty}\frac{E(u_{h_j},t)}{E(u_{h_j},0)}\ge G(t)$$

and complete the proof. □

4. Proof of Theorem 2.3. We follow the construction given by Colombini and Rauch in [1]. We first show a simple upper bound:

Proposition 4.1. *We have*

(4.1) $$\|\mathcal{R}(t,0)\|_{\mathcal{L}(\mathcal{H}\times L^2)}\le A^2(1+\delta(0))\times\exp\left(\frac{1}{2}\int_0^t\sup_{\substack{x\in\mathbf{R}^n\\ \xi\in\mathbf{R}^n}}\frac{|\partial_t a(s,x,\xi)|}{a(s,x,\xi)}ds\right)$$

for all $t\ge0$.

Proof. From (2.4) we have

$$\begin{aligned}
E(u,t)&=E(u,0)+\int_0^t\int\partial_t a(s,x,\nabla u)dxds\\
&\le E(u,0)\\
&\quad+\int_0^t\sup_{\substack{x\in\mathbf{R}^n\\ \zeta\in\mathbf{C}^n}}\frac{|\partial_t a(s,x,\zeta)|}{a(s,x,\zeta)}E(u,s)ds,
\end{aligned}$$

where

$$a(t,x,\zeta)=\sum_{i,j=1}^n a_{i,j}(t,x)\zeta_i\overline{\zeta_j}.$$

Using Gronwall's inequality, we get

$$E(u,t)\le E(u,0)\exp\left(\int_0^t\sup_{\substack{x\in\mathbf{R}^n\\ \zeta\in\mathbf{C}^n}}\frac{|\partial_t a(s,x,\zeta)|}{a(s,x,\zeta)}ds\right).$$

Noting

$$\sup_{\substack{x\in\mathbf{R}^n\\ \zeta\in\mathbf{C}^n}}\frac{|\partial_t a(s,x,\zeta)|}{a(s,x,\zeta)}=\sup_{\substack{x\in\mathbf{R}^n\\ \xi\in\mathbf{R}^n}}\frac{|\partial_t a(s,x,\xi)|}{a(s,x,\xi)}$$

and (2.2), we obtain (4.1). □

Proof of Theorem 2.3. Let $n=2$ and $\delta(t)$ be an arbitrary smooth nonnegative function. Using the standard identification $\mathbf{C}\ni u+iv\mapsto(u,v)\in\mathbf{R}^2$ of $\mathbf{R}^2$ with the complex plane, we write

$$x = re^{i\theta}, \quad \xi = \rho e^{i\phi}.$$

Let $f(\theta)$ be a smooth 2π periodic function verifying

$$f(0) = 0, \quad f'(0) = 1, \quad \sup_{\theta\in\mathbf{R}} |f| \le \frac{1}{2}, \quad \sup_{\theta\in\mathbf{R}} |f'| \le 1.$$

For example

$$f(\theta) = \frac{1}{2}\sin(2\theta)$$

satisfies these requirements. We define $\theta(t)$ and $\phi(t)$ by the solutions to

$$\begin{cases} \dfrac{d\theta}{dt}(t) = 1 + \delta(t), & \theta(0) = \dfrac{\pi}{2}, \\ \dfrac{d\phi}{dt}(t) = 1 + \delta(t), & \phi(0) = 0. \end{cases}$$

Let $\chi(r)$ be a smooth cut-off function such that $0 \le \chi(r) \le 1$ and

$$\chi(r) \equiv \begin{cases} 1 & \text{near } r = 1, \\ 0 & r \le 1/2 \text{ or } r \ge 2. \end{cases}$$

Let us define

$$\sqrt{\tilde{a}(t,r,\theta)} = \chi(r)e^{r-1}(1 + \delta(t) - 2\delta(t)f(\theta - \theta(t))) + 1 - \chi(r)$$

then $\tilde{a} \in C^\infty([0,\infty) \times \mathbf{R}^2)$ and (2.8) holds. Moreover, $r(t) \equiv 1, \theta(t), \phi(t)$ and $\rho(t) = \exp(2\int_0^t \delta(s)ds)$ satisfy the canonical equation with Hamiltonian $-\sqrt{\tilde{a}}$:

$$\text{(4.2)} \quad \begin{cases} \dfrac{dr}{dt} = -\sqrt{\tilde{a}}\cos(\theta - \phi), \\ \dfrac{d\theta}{dt} = \dfrac{1}{r}\sqrt{\tilde{a}}\sin(\theta - \phi), \\ \dfrac{d\phi}{dt} = -\dfrac{\partial\sqrt{\tilde{a}}}{\partial r}\sin(\phi - \theta) + \dfrac{1}{r}\dfrac{\partial\sqrt{\tilde{a}}}{\partial\theta}\cos(\phi - \theta), \\ \dfrac{d\rho}{dt} = \rho\left(\dfrac{\partial\sqrt{\tilde{a}}}{\partial r}\cos(\phi - \theta) + \dfrac{1}{r}\dfrac{\partial\sqrt{\tilde{a}}}{\partial\theta}\sin(\phi - \theta)\right). \end{cases}$$

Hence

$$X(t) = r(t)e^{i\theta(t)}, \quad \Xi(t) = \rho(t)e^{i\phi(t)}$$

are solutions to (2.5) and satisfy (2.6). Furthermore, we have

$$\sqrt{\frac{|\Xi(t)|}{|\Xi(0)|}} = \exp\left(\int_0^t \delta(s)ds\right).$$

Thus we can apply Corollary 2.2 and get (2.9).

Next we prove the latter assertion. Let $\delta(t)$ satisfy (2.10) and $\varepsilon > 0$. And let M be a large integer depending on $\varepsilon > 0$ and $f(\theta)$ a smooth 2π periodic function satisfying

$$f(0) = 0, \quad f'(0) = 1, \quad \sup_{\theta\in\mathbf{R}} |f| \le \frac{1}{M}, \quad \sup_{\theta\in\mathbf{R}} |f'| \le 1,$$

for example

$$f(\theta) = \frac{1}{M}\sin(M\theta).$$

Take $\theta(t), \phi(t)$ and $\chi(r)$ defined as above. We define

$$\sqrt{\tilde{a}(t,r,\theta)} = \chi(r)e^{r-1}(\delta(t) - 2\delta(t)f(\theta - \theta(t) + t)) + 1 - \chi(r).$$

Then $\tilde{a} \in C^\infty([0,\infty) \times \mathbf{R}^2)$ and (2.8) holds. Note that we can replace $(1 + \delta(t))$ by $\delta(t)$ in the condition (2.8). Furthermore, in this case $r(t) \equiv 1, \theta(t) - t, \phi(t) - t, \rho(t) = \exp(2\int_0^t \delta(s)ds)$ also satisfy (4.2). And then we can apply Corollary 2.2 and get the lower estimate of (2.11). To give the upper estimate of (2.11), it suffices to estimate

$$\sup_{\substack{x\in\mathbf{R}^n \\ \xi\in\mathbf{R}^n}} \frac{|\partial_t a(t,x,\xi)|}{a(t,x,\xi)}$$

from Proposition 4.1. We first obtain

$$\text{(4.3)} \quad \begin{aligned} \frac{|\partial_t\tilde{a}|}{\tilde{a}} &= 2\frac{|\partial_t\sqrt{\tilde{a}}|}{\sqrt{\tilde{a}}} \\ &\le 2\frac{(1 + 2/M)|\delta'| + 2\delta^2}{(1 - 2/M)\delta} \\ &= 2\frac{(1 + 2/M)|\delta'|}{(1 - 2/M)\delta} + \frac{4}{1 - 2/M}\delta. \end{aligned}$$

By using assumption (2.10), there exists $T_1 = T_1(M) > 0$ such that for all $t \ge T_1$

$$2\frac{(1 + 2/M)|\delta'(t)|}{(1 - 2/M)\delta(t)} \le \frac{1}{M}\delta(t).$$

Hence we have

$$\begin{aligned} &\exp\left(\int_{T_1}^t 2\frac{(1 + 2/M)|\delta'(s)|}{(1 - 2/M)\delta(s)} + \frac{4}{1 - 2/M}\delta(s)ds\right) \\ &\quad \le \exp\left(\left(\frac{1}{M} + \frac{4}{1 - 2/M}\right)\int_{T_1}^t \delta(s)ds\right) \end{aligned}$$

for all $t \ge T_1$. Therefore we put M_1 so that

$$\frac{1}{M_1} + \frac{4}{1 - 2/M_1} \le 4 + 2\varepsilon,$$

then

$$\|\mathcal{R}(t,T_1)\|_{\mathcal{L}(\mathcal{H}\times L^2)} \le A^2\delta(T_1)\exp\left((2 + \varepsilon)\int_{T_1}^t \delta(s)ds\right)$$

for $\tilde{a}$ defined from M which larger than M_1. From this and a standard energy inequality we have

$$\|\mathcal{R}(t,0)\|_{\mathcal{L}(\mathcal{H}\times L^2)} \leq C_0 \exp\left((2+\varepsilon)\int_0^t \delta(s)ds\right)$$

for $t \geq 0$ with some constant $C_0 = C_0(\varepsilon, M) > 0$. Thus we get (2.11).

Finally we prove (2.12). We can easily prove the lower estimate of (2.12) by modifying Lemma 2.1. To prove the upper estimate, following Doi-Nishitani-Ueda [2], we consider a modified energy

$$\tilde{E}(t) = E(t) + \beta(t)\,\mathrm{Re}(\partial_t u, u) + \gamma(t)\|u\|^2.$$

Here $\beta(t)$ and $\gamma(t)$ are chosen later. We define $\alpha(t)$ by the right hand side of (4.3). We obtain

$$\begin{aligned} \frac{d}{dt}\tilde{E}(t) \leq{}& \beta\tilde{E}(t) \\ &+ (\alpha - 2\beta)\int \tilde{a}|\nabla u|^2 dx \\ &+ (\beta' + 2\gamma - \beta^2)\,\mathrm{Re}(\partial_t u, u) \\ &+ (\gamma' - \beta\gamma)\|u\|^2. \end{aligned} \tag{4.4}$$

Now we put

$$\beta = \frac{2}{1-2/M}\,\delta, \quad \gamma = \frac{1}{2}(\beta^2 - \beta')$$

then we have

$$\beta' + 2\gamma - \beta^2 = 0,$$
$$\gamma' - \beta\gamma = \frac{1}{2}(3\beta\beta' - \beta'' - \beta^3).$$

We shall estimate $E(t)$ by $\tilde{E}(t)$. From the definition of $\beta(t)$ and the assumption (2.10), there exists $T_2 = T_2(\varepsilon, M) > 0, c_1 > 0$ such that

$$3\beta\beta' - \beta'' - \beta^3 \leq 0,$$
$$\frac{1}{3}\beta^2 - \beta' \geq c_1$$

for all $t \geq T_2$. By using the Schwarz inequality, there is a constant $c_2 > 0$ such that

$$\tilde{E} \geq \frac{1}{4}E + c_1\|u\|^2 \geq c_2\|(u, \partial_t u)(t)\|^2_{H^1\times L^2}.$$

In particular, we have

$$E(t) \leq 4\tilde{E}(t).$$

From this and (4.4) we obtain

$$\frac{d}{dt}\tilde{E}(u,t) \leq \left(\beta + 8\frac{(1+2/M)|\delta'|}{(1-2/M)\delta}\right)\tilde{E}(u,t)$$

for all $t \geq T_2$. Using the assumption (2.10) again, we get

$$8\frac{(1+2/M)|\delta'(t)|}{(1-2/M)\delta(t)} \leq \frac{1}{M}\delta(t)$$

for $t \geq T_3$ with sufficiently large T_3. Thus we have

$$\tilde{E}(t) \leq \tilde{E}(T_3)\exp\left(\left(\frac{2}{1-2/M} + \frac{1}{M}\right)\int_{T_3}^t \delta(s)ds\right).$$

Now we take M_2 so that

$$\frac{2}{1-2/M_2} + \frac{1}{M_2} \leq 2 + 2\varepsilon$$

then for any $M \geq M_2$ we have

$$\tilde{E}(t) \leq \tilde{E}(T_3)\exp\left((2+2\varepsilon)\int_{T_3}^t \delta(s)ds\right)$$

while $t \geq T_3$. From this and a standard energy estimate, it follows that

$$\|\mathcal{R}(t,0)\|_{\mathcal{L}(H^1\times L^2)} \leq C_2\exp\left((1+\varepsilon)\int_0^t \delta(s)ds\right)$$

for all $t \geq 0$ with some constant $C_2 = C_2(\varepsilon, M) > 0$. Thus, we finish the proof. □

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