Symmetric solutions for the two dimensional degenerate Garnier system G(5)

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(Communicated by Masaki KASHIWARA, M.J.A., June 13, 2011)

Abstract: We classify the symmetric solutions for the two dimensional degenerate Garnier system G(5) which is a generalization of the second Painlevé equation. We calculate the linear monodromy of the symmetric solutions explicitly.

Key words: Painlevé equation; Garnier system; monodromy data.

1. Introduction. We have studied special solutions with generic values of parameters for the fourth, fifth, sixth and third Painlevé equations, for which the monodromy data of the associated linear equation (we call linear monodromy) can be calculated explicitly [3–6].

The Garnier system was derived by R. Garnier (1912) as an extension of the sixth Painlevé equation [1,2]. The original Garnier system has n-variables and expressed in a system of nonlinear partial differential equations, whose dimension of the solution space is $2n$. There are few research for special solutions to the Garnier system compared with the Painlevé equations. We study the Garnier transcendents by applying the same method to the two dimensional Garnier system first, which we have used for the Painlevé equations above. Some new discovery is expected by viewing Painlevé equations from the Garnier system.

Two dimensional Garnier system has the following degeneration diagram similar to the Painlevé equations:

$$
\begin{array}{ccccccc}
\mathrm{G}(11111) \to & \mathrm{G}(1112) \to & \mathrm{G}(122) \to & \mathrm{G}(23) & & & \\
\swarrow & \swarrow & \swarrow & & & & \\
\mathrm{G}(113) \to & \mathrm{G}(14) \to & \mathrm{G}(5) \to & \mathrm{G}(9/2). & & &
\end{array}
$$

(The degeneration from G(113) to G(23) also exists.)

Numbers in the bracket ( ) represents a partition of 5 and 1 represents the regular singular point and $r+1$ represents an irregular singular point of Poincaré rank r. The two dimensional Garnier system G(11111), which is the extension of the sixth Painlevé equation, degenerates step by step to the two dimensional degenerate Garnier system G(9/2) which is the extension of the first Painlevé equation.

In this paper, we classify the symmetric solutions (See section 3) to the system G(5) with generic values of parameters, for which we calculate the linear monodromy of the equation (2.1) $\{M_\infty = S_1S_2\cdots S_8e^{2\pi i T_\infty}\}$ explicitly. In order to find out the symmetric solutions, we apply the same method to the system G(5), by which A. V. Kitaev discovered the symmetric solutions to the first and second Painlevé equations [8]. For the classification of the symmetric solutions, we use Prof. M. Suzuki's paper [10] and a single equation (2.10) for q_2. M. Suzuki constructed the space of the initial conditions of the two dimensional Garnier and its degenerate system. The space for the system G(5) consists of five charts which are glued by the symplectic transformation each other.

From the system G(5), we derive a single equation (2.10) for q_2, by solving which we have the order of pole and number of special solutions. On the other hand, we have special solutions on the each chart which consists of the space of the initial conditions. Both solutions coincide with each other and we confirmed that any other symmetric solution does not exist. We have five symmetric(meromorphic) solutions around the origin. We calculated the linear monodromy for the two solutions of them explicitly.

2. The two dimensional degenerate Garnier system $\mathcal{G}_2(5)$ and $\mathcal{H}_2(5)$. The two dimensional degenerate Garnier system $\mathcal{G}_2\{K_1, K_2,$

2000 Mathematics Subject Classification. Primary 34M55; Secondary 33C15.

doi: 10.3792/pjaa.87.114

$\lambda_1, \lambda_2, \mu_1, \mu_2, t_1, t_2\}(5)$ is derived as the extension of the second Painlevé equation by the isomonodromic deformation of the second kind, non Fuchsian ordinary differential equation, which has one irregular singular point of Poincaré rank 4 at $x = \infty$ on the Riemann sphere:

$$\begin{aligned}(2.1)\qquad &\frac{d^2y}{dx^2} - \left[2x^3 + 2t_1x + t_2 + \frac{1}{x-\lambda_1}\right.\\ &\left.+\frac{1}{x-\lambda_2}\right]\frac{dy}{dx} - \left[2\nu x^2 + 2K_2x\right.\\ &\left.+2K_1 - \frac{\mu_1}{x-\lambda_1} - \frac{\mu_2}{x-\lambda_2}\right]y = 0,\end{aligned}$$

where K_1 and K_2 are Hamiltonians and λ_1, λ_2, μ_1 and μ_2 are the Garnier transcendents and t_1 and t_2 are deformation parameters. The Riemann scheme of (2.1) is

$$P\begin{pmatrix} x=\lambda_1 & x=\lambda_2 & \overbrace{\quad x=\infty \quad}^{} & \\ 0 & 0 & 0\;\;0\;\;0\;\;0\;\;\;\nu & \\ 2 & 2 & \frac{1}{2}\;\;0\;\;t_1\;\;t_2\;\;1-\nu & ;x \end{pmatrix}.$$

$\mathcal{G}_2$ has movable algebraic branch points and Hamiltonian structure expressed in rational functions [7]:

$$\begin{aligned}K_1 &= -\sum_{k=1,2}\frac{P(\lambda_k)}{\Lambda'(\lambda_k)}\left[\mu_k^2 - \left(2\lambda_k^3 + 2t_1\lambda_k\right.\right.\\ &\quad\left.\left.+t_2 + \frac{1}{P(\lambda_k)}\right)\mu_k - 2\nu\lambda_k^2\right],\\ K_2 &= \sum_{k=1,2}\frac{1}{2\Lambda'(\lambda_k)}\left[\mu_k^2 - (2\lambda_k^3 + 2t_1\lambda_k\right.\\ &\quad\left.+t_2)\mu_k - 2\nu\lambda_k^2\right],\\ P(x) &= x - \lambda_1 - \lambda_2,\ \Lambda(x) = \prod_{k=1,2}(x-\lambda_k).\end{aligned}$$

We have the two dimensional degenerate Garnier system $\mathcal{H}_2\{H_1, H_2, q_1, q_2, p_1, p_2, s_1, s_2\}(5)$ by the canonical transformations:

$$\begin{aligned}&s_1 = \frac{t_1}{2},\quad s_2 = \frac{t_2}{2},\quad q_1 = \lambda_1\lambda_2 - \frac{t_2}{2},\\ &q_2 = -(\lambda_1+\lambda_2),\quad \mu_i = \sum_{j=1}^{2}\frac{q_jp_j}{\lambda_i - t_j},\\ &H_i = -(t_i-1)^2\left(K_i + \sum_{j=1}^{2}p_j\frac{\partial q_j}{\partial t_i}\right)(i=1,2),\end{aligned}$$

$$\begin{aligned}&\sum_{i=1}^{2}(dp_i\wedge dq_i - dH_i\wedge ds_i)\\ &\qquad = \sum_{i=1}^{2}(d\mu_i\wedge d\lambda_i - dK_i\wedge dt_i),\end{aligned}$$

where H_1 and H_2 are Hamiltonians and q_1, q_2, p_1 and p_2 are Garnier transcendents and s_1 and s_2 are the deformation parameters.

$\mathcal{H}_2$ has the Painlevé property and the polynomial Hamiltonian structure [7,9]. Hamiltonians H_1 and H_2 are given as follows:

$$\begin{aligned}H_1 &= (q_2^2 - q_1 - s_1)p_1^2 + 2q_2p_1p_2 + p_2^2\\ &\quad + 2(q_1^2 - s_1^2 + s_2q_2)p_1 + 2(q_1q_2\\ &\quad + s_1q_2 + s_2)p_2 + 2\nu q_1,\\ H_2 &= q_2p_1^2 + 2p_1p_2 + 2(q_1q_2 + s_1q_2\\ &\quad + s_2)p_1 + 2(q_2^2 - q_1 + s_1)p_2\\ &\quad + 2\nu q_2,\qquad (2\nu = 2\alpha + 1),\end{aligned}$$

from which we have the polynomial Hamiltonian system $\mathcal{H}_2(5)$:

$$\begin{aligned}(2.2)\qquad &\frac{\partial q_1}{\partial s_1} = 2\big(q_2^2 - q_1 - s_1\big)p_1\\ &\qquad + 2q_2p_2 + 2(q_1^2 - s_1^2 + s_2q_2),\\ (2.3)\qquad &\frac{\partial q_2}{\partial s_1} = 2q_2p_1 + 2p_2\\ &\qquad + 2(q_1q_2 + s_1q_2 + s_2),\\ (2.4)\qquad &-\frac{\partial p_1}{\partial s_1} = -p_1^2 + 4q_1p_1 + 2q_2p_2 + 2\nu,\\ (2.5)\qquad &-\frac{\partial p_2}{\partial s_1} = 2q_2p_1^2 + 2p_1p_2 + 2s_2p_1\\ &\qquad + 2(q_1 + s_1)p_2,\\ (2.6)\qquad &\frac{\partial q_1}{\partial s_2} = 2q_2p_1 + 2p_2\\ &\qquad + 2(q_1q_2 + s_1q_2 + s_2),\\ (2.7)\qquad &\frac{\partial q_2}{\partial s_2} = 2p_1 + 2(q_2^2 - q_1 + s_1),\\ (2.8)\qquad &-\frac{\partial p_1}{\partial s_2} = 2q_2p_1 - 2p_2,\\ (2.9)\qquad &-\frac{\partial p_2}{\partial s_2} = p_1^2 + 2(q_1 + s_1)p_1 + 4q_2p_2 + 2\nu.\end{aligned}$$

The Hamiltonian system $\mathcal{H}_2(5)$ has a holomorphic solution around the origin for the given initial condition by Cauchy's theorem. We derive the single equation for q_2 of the Hamiltonian system $\mathcal{H}_2(5)$ so that we investigate whether $\mathcal{H}_2(5)$ has a solution with a pole. Eliminating q_1, p_1 and p_2 from (2.6), (2.7), (2.8) and (2.9), we have the following partial differential equation of the fourth order

$$(2.10)\quad 2q_2^2\frac{\partial^4 q_2}{\partial s_2^4} - 4q_2\frac{\partial q_2}{\partial s_2}\frac{\partial^3 q_2}{\partial s_2^3} + 4\left(\frac{\partial q_2}{\partial s_2}\right)^2\frac{\partial^2 q_2}{\partial s_2^2} - 40q_2^4\frac{\partial^2 q_2}{\partial s_2^2} - 16s_2q_2\frac{\partial^2 q_2}{\partial s_2^2} - 3q_2\left(\frac{\partial^2 q_2}{\partial s_2^2}\right)^2 - 16q_2\frac{\partial q_2}{\partial s_2} - 20q_2^3\left(\frac{\partial q_2}{\partial s_2}\right)^2 + 16s_2\left(\frac{\partial q_2}{\partial s_2}\right)^2 + 80q_2^7 + 192s_1q_2^5 + 64s_2q_2^4 + (64s_1^2 - 96\nu + 48)q_2^3 - 16s_2^2q_2 = 0.$$

Equation (2.10) has the formal solutions in the following form:

$$(2.11)\quad q_2 = \sum_{j=-1}^{\infty} b_j(s_1)s_2^j,\ b_{-1} = 0, \pm\frac{1}{2}, \pm 1,$$

and the solutions with a pole of order 1 around the origin. It is necessary that $b_{-1} = 0$ or $\pm\frac{1}{2}$ for the solution (2.11) when $q_2(s_1, s_2)$ has a pole along $s_2 = 0$. If $b_{-1} = \pm 1$, q_2 has a pole divisor which passes through the origin.

3. Symmetric solutions around the origin. In this section, we recall the symmetric solution to the second Painlevé equation which was given by A. V. Kitaev [8]. He showed that the second Painlevé equation

$$P_{II}: \quad \frac{d^2y}{dt^2} = 2y^3 + ty + \alpha$$

is invariant by the transformation (we call the symmetric transformation):

$$y \to \omega y, \quad t \to \omega^2 t\ (\omega^3 = 1)$$

and P_{II} has the solution:

$$y = \frac{\alpha}{2}t^2 + \frac{\alpha}{40}t^5 + \frac{10\alpha^3 + \alpha}{2280}t^8 + \cdots,$$

which is invariant by the symmetric transformation (we call the symmetric solution).

By applying this method to the system $\mathcal{H}_2(5)$, we obtain the following theorem.

Theorem 1. (1) *The Hamiltonian system $\mathcal{H}_2(5)$ is invariant by the symmetric transformation*:

$$(3.1)\quad q_1 \to \rho^2 q_1,\ p_1 \to \rho^2 p_1,\ q_2 \to \rho q_2,\ p_2 \to \rho^3 p_2 \quad s_1 \to \rho^2 s_1,\ s_2 \to \rho^3 s_2, \quad (\rho^4 = 1).$$

(2) *The Hamiltonian system $\mathcal{H}_2(5)$ has the following holomorphic and symmetric solution* (5-*a*) *around the origin*:

$$(5\text{-}a)\quad q_1 = \sum_{2i+3j\equiv 2 (mod\, 4)} a_{i,j}s_1^is_2^j = (1-2\nu)s_2^2 - \frac{2}{3}(1-2\nu)s_1^3 + 2(2\nu^2 - 2\nu + 1)s_1^2s_2^2 + \cdots,$$

$$p_1 = \sum_{2i+3j\equiv 2 (mod\, 4)} \tilde{a}_{i,j}s_1^is_2^j = -2\nu s_1 - 2\nu s_2^2 + \frac{4}{3}\nu^2 s_1^3 + \cdots,$$

$$q_2 = \sum_{2i+3j\equiv 1 (mod\, 4)} b_{i,j}s_1^is_2^j = 2(1-2\nu)s_1s_2 - \frac{2}{3}s_2^3 + \frac{4}{3}(2\nu^2 - 2\nu + 1)s_1^3s_2 + \cdots,$$

$$p_2 = \sum_{2i+3j\equiv 3 (mod\, 4)} \tilde{b}_{i,j}s_1^is_2^j = -2\nu s_2 + 4\nu(1-\nu)s_1^2s_2 + 8\nu(1-2\nu)s_1s_2^3 + \cdots.$$

Remark 2. Higher order expansion of this solution are determined recursively by the Hamiltonian system $\mathcal{H}_2(5)$.

M. Suzuki showed that the space of the initial conditions of the degenerate Garnier system G(5) consists of five charts which are glued each other by the following symplectic transformations [10]:

$$(b)\quad q_1 = \frac{\tilde{q}_1}{\tilde{q}_2},\ q_2 = \frac{1}{\tilde{q}_2},\ p_1 = \tilde{q}_2\tilde{p}_1,\quad p_2 = -\tilde{q}_2(\nu + \tilde{q}_1\tilde{p}_1 + \tilde{q}_2\tilde{p}_2),$$

$$(c)\quad \tilde{q}_1 = \hat{q}_1,\quad \tilde{q}_2 = \hat{q}_2,$$

$$\tilde{p}_1 = \frac{-2}{\tilde{q}_2^3} + \frac{2\tilde{q}_1}{\tilde{q}_2^2} - \frac{2s_1}{\tilde{q}_2} + \hat{p}_1,$$

$$\tilde{p}_2 = \frac{-2}{\tilde{q}_2^5} + \frac{6\tilde{q}_1}{\tilde{q}_2^4} - \frac{2\tilde{q}_1^2}{\tilde{q}_2^3} + \frac{2(s_1\tilde{q}_1 + s_2)}{\tilde{q}_2^2} - \frac{2\nu}{\tilde{q}_2} + \hat{p}_2,$$

$$(d)\quad q_1 = \frac{1}{\check{q}_1},\ q_2 = \frac{\check{q}_2}{\check{q}_1},\quad p_2 = \check{q}_1\check{p}_2,\quad p_1 = -\check{q}_1(\nu + \check{q}_1\check{p}_1 + \check{q}_2\check{p}_2),$$

$$(e)\quad \check{q}_1 = \bar{q}_1,\quad \check{q}_2 = \bar{q}_2,$$

$$\check{p}_1 = -\frac{2\check{q}_2^4}{\check{q}_1^5} + \frac{6\check{q}_2^2}{\check{q}_1^4} - \frac{2}{\check{q}_1^3} + \frac{2(s_2\check{q}_2 + s_1)}{\check{q}_1^2} - \frac{2\nu}{\check{q}_1} + \bar{p}_1,$$
$$\check{p}_2 = \frac{2\check{q}_2^3}{\check{q}_1^4} - \frac{4\check{q}_2}{\check{q}_1^3} - \frac{2s_2}{\check{q}_1} + \bar{p}_2,$$
$$(e) \leftarrow (d) \leftarrow (a) \rightarrow (b) \rightarrow (c),$$

where (a) represents the base chart on which the solution (5-a) exists.

Remark 3. (1) By these symplectic transformations, the symmetric transformation (3.1) is kept with a different weight.

(2) On each chart, there is one holomorphic and symmetric solution exists.

By the symplectic transformation (b), we have the new polynomial Hamiltonians and Hamiltonian system, where we have the following holomorphic solution:

$$\tilde{q}_1 = \sum_{2i+3j\equiv 1 (mod\, 4)} a_{i,j} s_1^i s_2^j = 2s_1s_2 - \frac{4}{3}s_2^3 + \frac{8}{3}(2+3\nu)s_1^2s_2^3 + \cdots,$$
$$\tilde{p}_1 = \sum_{2i+3j\equiv 3 (mod\, 4)} a_{i,j} s_1^i s_2^j = -2\nu s_2 + \frac{32}{3}\nu s_1 s_2^3 + \cdots,$$
$$\tilde{q}_2 = \sum_{2i+3j\equiv 3 (mod\, 4)} a_{i,j} s_1^i s_2^j = -2s_2 - \frac{16}{3}s_1s_2^3 + \frac{16}{15}(1-6\nu)s_2^5 + \cdots,$$
$$\tilde{p}_2 = \sum_{2i+3j\equiv 1 (mod\, 4)} a_{i,j} s_1^i s_2^j = 2\nu s_1 s_2 + \frac{4}{3}\nu(1+3\nu)s_2^3 - \frac{16}{3}\nu(2+5\nu)s_1^2s_2^3 + \cdots.$$

By making inverse transformation of (b), we have the following solution (5-b):

$$(5\text{-}b) \quad q_1 = -s_1 + \frac{2}{3}s_2^2 - 4\nu s_1^2 s_2^2 + \cdots,$$
$$p_1 = 4\nu s_2^2 - \frac{32}{3}\nu s_1 s_2^4 + \cdots,$$
$$q_2 = -\frac{1}{2s_2} + \frac{4}{3}s_1s_2 - \frac{4}{5}\left(\frac{1}{3} - 2\nu\right)s_2^3 + \cdots,$$
$$p_2 = 2\nu s_2 - \frac{32}{3}\nu s_1 s_2^3 + \cdots.$$

By the similar way, we have the following solutions (5-c), (5-d) and (5-e):

$$(5\text{-}c) \ q_1 = -s_1 - \frac{2}{3}s_2^2 + \cdots,$$
$$p_1 = -\frac{1}{2s_2^2} - \frac{4}{3}s_1 - \frac{4}{5}\nu s_2^2 + \cdots,$$
$$q_2 = \frac{1}{2s_2} - \frac{4}{3}s_1s_2 + \left(\frac{4}{3} - \frac{8}{5}\nu\right)s_2^3 + \cdots,$$
$$p_2 = \frac{1}{4s_2^3} - 2\left(\frac{1}{3} + \frac{\nu}{5}\right)s_2 + \cdots.$$

(5-d)

$$q_1 = \frac{1}{-2s_1 + 2s_2^2 - \frac{8}{3}\nu s_1^3 - 4s_1^2s_2^2 + \cdots},$$
$$p_1 = 2\nu s_1 - 2\nu s_2^2 - \frac{4}{3}\nu^2 s_1^3 + \cdots,$$
$$q_2 = \frac{-s_2 - 2(1+2\nu)s_1^2 s_2 - 2s_1s_2^3 + \cdots}{-s_1 + s_2^2 - \frac{4}{3}\nu s_1^3 - 2s_1^2s_2^2 + \cdots},$$
$$p_2 = -4\nu(\nu+1)s_1^2s_2 - 4\nu\left(\nu + \frac{2}{3}\right)s_1s_2^3 + \cdots.$$

(5-e)

$$q_1 = \frac{1}{2s_1 + 2s_2^2 - \frac{8}{3}(1-\nu)s_1^3 + \cdots},$$
$$p_1 = \frac{-[s_1 - s_2^2 - \frac{4}{3}(1-\nu)s_1^3 + \cdots]}{[s_1 + s_2^2 - \frac{4}{3}(1-\nu)s_1^3 + \cdots]^2} - 2(2-\nu)s_1 - 2(1-\nu)s_2^2 + \cdots,$$
$$q_2 = \frac{s_2 - 2(3-2\nu)s_1^2s_2 + 2s_1s_2^3 + \cdots}{s_1 + s_2^2 - \frac{4}{3}(1-\nu)s_1^3 + \cdots},$$
$$p_2 = \frac{2[s_2 - 2(3-2\nu)s_1^2s_2 + \cdots]}{[s_1 + s_2^2 - \frac{4}{3}(1-\nu)s_1^3 + \cdots]^3} \times \left[-s_1 + \frac{4}{3}(1-\nu)s_1^3 + \cdots\right] - 2s_2 - 4(1-\nu)(3-\nu)s_1^2s_2 + \cdots.$$

Remark 4. (1) Putting $s_1 = 0$, q_2 has the coefficient ∓ 1 of s_2^{-1} in the solutions (5-d) and (5-e).

(2) The Hamiltonian system $\mathcal{H}_2(5)$ has three solutions in the form of (2.11) with $b_{-1} = 0, \pm 1/2$ and two solutions in the form of the solutions (5-d) and (5-e) with $b_{-1} = \pm 1$.

(3) The number of the residue at the pole of the solution of the single equation (2.10) gives the number of charts which consists of the space of the initial conditions.

We have the following theorem:

Theorem 5. *The Hamiltonian system* $\mathcal{H}_2(5)$ *has five symmetric solutions* (5-a),(5-b),(5-c),(5-d)

and (5-*e*) *and any other symmetric solution does not exist around the origin.*

4. The linear monodromy. In this section, we calculate the linear monodromy for the solution (5-*a*) and (5-*d*).

For the solution (5-*a*), the linear equation becomes

$$\frac{d^2\psi_1}{dx^2} - \left(\frac{2}{x} + 2x^3\right)\frac{d\psi_1}{dx} - 2\nu x^2\psi_1 = 0,$$

whose Poincaré rank at $x = \infty$ is 4. This is reduced to Kummer's equation:

$$\frac{d^2\psi_1}{d\xi^2} + \left(\frac{1}{4\xi} - 1\right)\frac{d\psi_1}{d\xi} - \frac{\nu}{4\xi}\psi_1 = 0, \quad (x^4 = 2\xi).$$

For the solution (5-*d*), the linear equation becomes

$$\frac{d^2\psi_1}{dx^2} - 2x^3\frac{d\psi_1}{dx} - 2\nu x^2\psi_1 = 0,$$

whose Poincaré rank at $x = \infty$ is 4. This is also reduced to Kummer's equation:

$$\frac{d^2\psi_1}{d\xi^2} + \left(\frac{3}{4\xi} - 1\right)\frac{d\psi_1}{d\xi} - \frac{\nu}{4\xi}\psi_1 = 0, \quad (x^4 = 2\xi).$$

We have the next theorem.

Theorem 6. *The two dimensional degenerate Garnier system* $G(5)$ *has the following linear monodromy.*

(1) *For the solution* (5-*a*):

$$e^{2\pi i T_\infty} = \begin{pmatrix} e^{2\pi i\nu} & 0 \\ 0 & e^{-2\pi i\nu} \end{pmatrix},$$

$$S_{2k} = \begin{pmatrix} 1 & \dfrac{-2\pi i e^{-\frac{\pi}{2}ik(1-2\nu)}}{\Gamma(1-\frac{\nu}{4})\Gamma(\frac{1}{4}-\frac{\nu}{4})} \\ 0 & 1 \end{pmatrix},$$

$$S_{2k-1} = \begin{pmatrix} 1 & 0 \\ \dfrac{-2\pi i e^{\frac{\pi}{4}i(2k-1)(1-2\nu)}}{\Gamma(\frac{\nu}{4})\Gamma(\frac{3}{4}+\frac{\nu}{4})} & 1 \end{pmatrix},$$

(2) *For the solution* (5-*d*):

$$e^{2\pi i T_\infty} = \begin{pmatrix} e^{2\pi i\nu} & 0 \\ 0 & e^{-2\pi i\nu} \end{pmatrix},$$

$$S_{2k} = \begin{pmatrix} 1 & \dfrac{-2\pi i e^{-\frac{\pi}{2}ik(3-2\nu)}}{\Gamma(1-\frac{\nu}{4})\Gamma(\frac{3}{4}-\frac{\nu}{4})} \\ 0 & 1 \end{pmatrix},$$

$$S_{2k-1} = \begin{pmatrix} 1 & 0 \\ \dfrac{-2\pi i e^{\frac{\pi}{4}i(2k-1)(3-2\nu)}}{\Gamma(\frac{\nu}{4})\Gamma(\frac{1}{4}+\frac{\nu}{4})} & 1 \end{pmatrix},$$

$$(k = 1, 2, 3, 4), \quad \prod_{j=1}^{8} S_j e^{2\pi i T_\infty} = I_2,$$

where $e^{2\pi i T_\infty}$ *is the formal monodromy and* S_j $(j = 1, 2, \cdots, 8)$ *is the Stokes matrices at* $x = \infty$.

Remark 7. For the solution (5-*b*), (5-*c*) and (5-e), the linear monodromy cannot be calculated explicitly since the coefficient of the linear equation (2.1) becomes ∞.

Acknowledgements. The author would like to express his sincere gratitude to Prof. Yousuke Ohyama for his constant guidance over time and useful suggestions to complete this work. This work was supported by JSPS 22540237 and the Mitsubishi Foundation.

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