Construction of 3-Hilbert class field of certain imaginary quadratic fields

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Abstract: In this paper, we give an explicit description of 3-Hilbert class field of certain imaginary quadratic fields.

Key words: Hilbert class field; anti-cyclotomic extension; Kummer extension.

1. Introduction. Let k be an imaginary quadratic field, and L an abelian extension of k. L is called an anti-cyclotomic extension of k if it is Galois over $\mathbf{Q}$, and $Gal(k/\mathbf{Q})$ acts on $Gal(L/k)$ by -1. For each prime number p, the compositum K of all $\mathbf{Z}_p$-extensions over k becomes a ${\mathbf{Z}_p}^2$-extension, and K is the compositum of the cyclotomic $\mathbf{Z}_p$-extension and the anti-cyclotomic $\mathbf{Z}_p$-extension of k. In the paper [2], using Kummer theory and class field theory, we constructed the first layer k_1^a of the anti-cyclotomic $\mathbf{Z}_p$-extension of an imaginary quadratic field whose class number is not divisible by p. In this paper, we apply the same method as in [2] to construct 3-Hilbert class field of certain imaginary quadratic fields.

2. Proof of theorems. We begin this section by explaining how to construct a cyclic extension M_p of prime degree p of an imaginary quadratic field k, which is unramified outside p over k and $Gal(M_p/\mathbf{Q}) \simeq D_p$, the dihedral group of order $2p$. Throughout this section, we denote by H_k, h_k, A_k the p-part of Hilbert class field, the p-class number, and p-part of ideal class group of k, respectively.

Let k be an imaginary quadratic field and ζ_p a primitive p-th root of unity. We denote $k_z = k(\zeta_p)$ and let σ, τ with $\sigma(\zeta_p) = {\zeta_p}^t$ be generators of $Gal(k_z/k), Gal(k_z/\mathbf{Q}(\zeta_p))$, respectively. Assume that $p \neq 2$ and $k \neq \mathbf{Q}(\sqrt{-3})$ if $p = 3$. Then we have the following theorem which is a refinement of [2, Theorem 1].

Theorem 1 (See [2, Theorem 1]). *Let X' be a vector space over a finite field F_p with a basis $\{x_1, \cdots, x_{p-1}\}$ and A be a linear map such that $Ax_i = x_{i+1}$ for $i = 1, \cdots, p-2$ and $Ax_{p-1} = x_1$. Let $x = \sum_i a_i x_i$ be an eigenvector of A corresponding to an eigenvalue t satisfying $\sigma(\zeta_p) = {\zeta_p}^t$. Let $k = \mathbf{Q}(\sqrt{-D})$ be an imaginary quadratic field. Assume that $\varepsilon = \tau(\epsilon)\epsilon^{-1}$ is not a p-power of a unit in k_z, where $\epsilon = \prod_i (\alpha)^{a_i \sigma^{i-1}}$ for some unit $\alpha \in k_z$. Then $k_z(\sqrt[p]{\varepsilon})$ contains a unique cyclic extension M_p of prime degree p of k, which is unramified outside p over k and $Gal(M_p/\mathbf{Q}) \simeq D_p$, and $M_p = k(\eta)$ where $\eta = Tr_{k_z(\sqrt[p]{\varepsilon})/M_p}(\sqrt[p]{\varepsilon})$.*

Proof of Theorem 1. We include here the proof of Theorem 1 given in [2] because we will use Theorem 1 to construct the 3-Hilbert class field of k. Write $L_z = k_z(\sqrt[p]{\varepsilon})$. Let $H = \langle \varepsilon \ mod (k_z^*)^p \rangle$ be the Kummer group for the Kummer extension L_z/k_z, and let $X = Gal(L_z/k_z)$. Then $Gal(k_z/\mathbf{Q})$ acts on H and X, and the Kummer pairing

$$H \times X \longrightarrow \mu_p$$

is a perfect $Gal(k_z/\mathbf{Q})$-equivariant pairing. Hence, by the construction of ε, σ and τ, and using the fact that $F_p[Gal(k_z/k)] \simeq X' : \sigma^{i-1} \rightarrow x_i$, one can easily see that $\sigma(\varepsilon) = \varepsilon^t \ mod (k_z^*)^p$ and $\tau(\varepsilon) = \varepsilon^{-1}$. Therefore the generators σ and τ act on X trivially and inversely, respectively. It follows that $Gal(L_z/k)$ is cyclic of order $(p-1)p$. Then there exists the unique intermediate field M_p of L_z/k with $[M_p : k] = p$, and the uniqueness of M_p asserts that $M_p/\mathbf{Q}$ is a Galois extension. It follows that $Gal(M_p/\mathbf{Q}) \simeq Gal(L_z/\mathbf{Q}(\zeta_p)) \simeq D_p$. Moreover, M_p/k is a cyclic extension of degree p unramified outside p, and by [1, Theorem 5.3.5] we conclude that $M_p = k(\eta)$ with $\eta = Tr_{L_z/M_p}(\sqrt[p]{\varepsilon})$.

The first layer of anti-cyclotomic $\mathbf{Z}_p$-extension of an imaginary quadratic field k may be or may not be contained in the p-Hilbert class field of k. The following theorem gives an answer for this question when $p = 3$.

Theorem 2 (See [3, Theorem 2]). *Let $d \not\equiv 3 \ mod\ 9$ be a squarefree positive integer, $k = \mathbf{Q}(\sqrt{-d})$*

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an imaginary quadratic field and K the compositum of all $\mathbf{Z}_3$-extensions over k. Then

$$H_k \cap K = k \Longleftrightarrow$$
$$rank_{\mathbf{Z}/3} A_{\mathbf{Q}(\sqrt{3d})} = rank_{\mathbf{Z}/3} A_{\mathbf{Q}(\sqrt{-d})}.$$

Remark 1. *It is well-known that*

$$rank_{\mathbf{Z}/3} A_{\mathbf{Q}(\sqrt{3d})} \leq rank_{\mathbf{Z}/3} A_{\mathbf{Q}(\sqrt{-d})} \leq rank_{\mathbf{Z}/3} A_{\mathbf{Q}(\sqrt{3d})} + 1.$$

Now we state the main theorem of this paper, and prove it by using the results discussed above.

Theorem 3. *Let $d \not\equiv 3$ mod 9 be a squarefree positive integer and $k = \mathbf{Q}(\sqrt{-d})$ be an imaginary quadratic field such that $h_k = 3$ and $h_{\mathbf{Q}(\sqrt{3d})} = 1$. Then*

$$H_k = k(\sqrt[3]{\alpha^2} + \sqrt[3]{\alpha^{-2}})$$

where α is the fundamental unit of $\mathbf{Q}(\sqrt{3d})$.

Proof of Theorem 3. Since the absolute norm $N\alpha = \alpha^{\sigma+1} = \alpha^{\tau+1} = \pm 1$, Note that $p = 3$ and $2+\sigma$ is the eigenvector for the eigenvalue $t = 2$, we have $\epsilon = \alpha^{2+\sigma} = \pm\alpha$ and hence $\varepsilon = \pm\alpha^{-2}$ is not cubic in k_z. Since $\sigma^{\sim} \in Gal(L_z/M_3)$ satisfies $\sigma^{\sim 2} = 1$, we have $\sqrt[3]{\varepsilon}^{\sigma^{\sim}} = \sqrt[3]{\varepsilon}^{-1}$ and therefore $\eta = \pm(\sqrt[3]{\alpha^2} + \sqrt[3]{\alpha^{-2}})$ and $M_3 = k(\eta)$. Let F be the maximal abelian p-extension of k unramified outside p. Then [4] class field theory shows that

$$Gal(F/H_k) \simeq \left(\prod_{\mathfrak{p}|p} U_{1,\mathfrak{p}}\right),$$

where $U_{1,\mathfrak{p}}$ is the local units of k which is congruent to 1 *mod* $\mathfrak{p}$. So in this case, by Theorem 2, $Gal(F/k) \simeq \mathbf{Z}_p^2$. Hence F, which is equal to K in this case, contains a unique D_p-extension k_1^a of $\mathbf{Q}$ (cf. [2, Lemma 1]). This completes the proof since M_p, H_k, and k_1^a are D_p-extensions of $\mathbf{Q}$ contained in F.

Let us give an example of Theorem 3.

Example 1. Let $k = \mathbf{Q}(\sqrt{-23})$ and $p = 3$. In this case, $h_k = 3$, $h_{\mathbf{Q}(\sqrt{69})} = 1$. We can take $t = 2$, and in this case the eigenvector of A is $2x_1 + x_2$. If we take $\alpha = 11 + 3\left(\frac{1+\sqrt{69}}{2}\right)$, then $\epsilon = \alpha^{2+\sigma} = \alpha$ and $\varepsilon = \epsilon^{\tau-1} = \left(11 + 3\left(\frac{1-\sqrt{69}}{2}\right)\right)^2$. Note that α is a fundamental unit of $\mathbf{Q}(\sqrt{69})$ and $\varepsilon = \left(11 + 3\left(\frac{1-\sqrt{69}}{2}\right)\right)^2 = \alpha^{-2}$. Hence ε is not a 3rd power in k_z. We can easily compute that $\eta = \sqrt[3]{\frac{623-75\sqrt{69}}{2}} + \sqrt[3]{\frac{623+75\sqrt{69}}{2}}$ and $irr(\eta, \mathbf{Q}(\sqrt{-23})) = x^3 - 3x - 623$. Therefore H_k is the splitting field of $x^3 - 3x - 623$ over $\mathbf{Q}(\sqrt{-23})$.

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