Note on mod p decompositions of gauge groups

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Abstract: We give fibrewise mod p decompositions of the adjoint bundle of a principal G-bundle P when the topological group G has mod p decompositions by automorphisms as in [5], which imply mod p decompositions of the gauge group of P.

Key words: Gauge group; mod p decomposition.

1. Introduction and statement of the result. We will always assume that spaces have the homotopy types of CW-complexes.

Let G be a connected topological group, and let P be a principal G-bundle over a space B. The gauge group of P, denoted $\mathcal{G}(P)$, is the topological group of G-equivariant self-maps of P covering the identity of B with the compact open topology, where the group structure is given by the composite of maps. For an action ρ of G on a space F, we denote by $P \times_\rho F$ the fibre bundle associated to P with the action ρ. In the special case that ρ is the adjoint action of G onto G itself, we put $\mathrm{ad}\, P = P \times_\rho G$ and call it the adjoint bundle of P. Note that $\mathrm{ad}\, P$ is a fibrewise topological group in the sense of [3]. Then if we denote the space of sections of a fibrewise space $E \to B$ by $\Gamma(E)$, we have that $\Gamma(\mathrm{ad}\, P)$ is a topological group. It is shown in [1] that there is a natural isomorphism of topological groups:

$$\mathcal{G}(P) \cong \Gamma(\mathrm{ad}\, P)$$

Thus a fibrewise decomposition of the adjoint bundle $\mathrm{ad}\, P$ yields a decomposition of the gauge group $\mathcal{G}(P)$.

The gauge group $\mathcal{G}(P)$, of course, inherits the structures of the topological group G. Then if we have a decomposition of G, $\mathcal{G}(P)$ may have a decomposition. In fact, Theriault [11] showed that mod p decompositions of G induce those of $\mathcal{G}(P)$ when the base space B is S^4. Other decompositions of gauge groups are discussed in [7] and [8]. The aim of this note is to produce a fibrewise mod p decomposition of the adjoint bundle $\mathrm{ad}\, P$ for yielding a mod p decomposition of the gauge group $\mathcal{G}(P)$ when G has a mod p decomposition by an automorphism as in [5].

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In order to state the result, we need some notation. Let $\mathbf{P}$ be a set of primes. We denote by $-_{\mathbf{P}}$ the localization away from $\mathbf{P}$ in the sense of Hilton, Mislin and Roitberg [6]. We also denote by $-^f_{\mathbf{P}}$ the fibrewise localization away from $\mathbf{P}$ in the sense of May [9].

Suppose G has an automorphism α with the subgroup of fixed points H. We define a map $\sigma : G/H \to G$ by

$$\sigma(gH) = g\alpha(g)^{-1}$$

for $g \in G$. We also define a map $\theta : H \times G/H \to G$ by

$$\theta(h, gH) = h \cdot \sigma(gH)$$

for $h \in H$ and $g \in G$. Let ρ be the action of H upon G/H defined by

$$\rho(h, gH) = hgH$$

for $h \in H$ and $g \in G$. Now we give the main theorem whose proof will be given in the next section where we also give some examples.

Theorem 1.1. *Let G, H, θ and ρ be as above. Suppose that the localized map $\theta_{\mathbf{P}}$ is a homotopy equivalence for some set of primes $\mathbf{P}$. Then there is a fibrewise homotopy equivalence:*

$$(\mathrm{ad}\, EG|_{BH})^f_{\mathbf{P}} \simeq_{BH} (\mathrm{ad}\, EH)^f_{\mathbf{P}} \times_{BH} (EH \times_\rho G/H)^f_{\mathbf{P}}$$

Let $E \to B$ be a fibration whose fibre is connected and nilpotent. It follows from the result of Møller [10] that the induced map $\Gamma(E) \to \Gamma(E^f_{\mathbf{P}})$ from the fibrewise localization $E \to E^f_{\mathbf{P}}$ is the localization $\Gamma(E) \to \Gamma(E)_{\mathbf{P}}$. Obviously, we have $\Gamma(E_1 \times_B E_2) \cong \Gamma(E_1) \times \Gamma(E_2)$ for fibrewise spaces E_1 and E_2 over B. Then we obtain:

Corollary 1.1. *Let G, H, θ and ρ be as in Theorem* 1.1. *Suppose that the localized map $\theta_{\mathbf{P}}$ is a*

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homotopy equivalence for some set of primes $\mathbf{P}$. *Then there is a homotopy equivalence*:

$$\mathcal{G}(EG|_{BH})_{\mathbf{P}} \simeq \mathcal{G}(EH)_{\mathbf{P}} \times \Gamma(EH \times_\rho G/H)_{\mathbf{P}}$$

2. Proof of Theorem 1.1 and examples. We first give a proof of Theorem 1.1. Let ad_H denote the adjoint action of H onto G. Then we have a commutative diagram

$$\begin{array}{ccc} H \times G/H & \xrightarrow{\ \rho\ } & G/H \\ {\scriptstyle 1\times\sigma}\downarrow & & \downarrow{\scriptstyle \sigma} \\ H \times G & \xrightarrow{\ \mathrm{ad}_H\ } & G \end{array}$$

which induces a fibrewise map τ fitting into the following commutative diagram of fibre sequences.

$$\begin{array}{ccc} H \times G/H & \xrightarrow{\qquad \theta \qquad} & G \\ \downarrow & & \downarrow \\ \mathrm{ad}\, EH \times_{BH} (EH \times_\rho G/H) & \xrightarrow{\ \tau\ } & \mathrm{ad}\, EG|_{BH} \\ \downarrow & & \downarrow \\ BH & = & BH \end{array}$$

Thus Theorem 1.1 follows from Dold's theorem together with the assumption that the localized map $\theta_{\mathbf{P}}$ is a homotopy equivalence.

Next, we give some examples to which we can apply Theorem 1.1 and Corollary 1.1. The following special gauge groups are of our main interesting. Let G be a connected simple Lie group. Then the principal G-bundle over S^4 is classified by $\pi_3(G) \cong \mathbf{Z}$.

Definition 2.1. We denote by $\mathcal{G}_k(G)$ the gauge group of principal G-bundle classified by $k \in \mathbf{Z} \cong \pi_3(G)$.

Example 2.1. Let G, H, p, d and α be as in Table I. Here the matrix J is $\begin{pmatrix} O & E_n \\ -E_n & O \end{pmatrix}$. Then each α is an automorphism of G with the subgroup of fixed points H. Note that the order of α equals p. In [5], Harris showed that the localized map $\theta_{\frac{1}{p}}$ is a homotopy equivalence, where $-_{\frac{1}{p}}$ stands for the localization away from the set of all primes but p, that is, inverting p. Then we can apply Theorem 1.1 and Corollary 1.1. Moreover, since the inclusion $H \to G$ induces d-multiplication in π_3, we have obtained:

Table I.

G	H	p	d	α
$\mathrm{SU}(2n+1)$	$\mathrm{SO}(2n+1)$	2	2	complex conjugation
$\mathrm{SU}(2n)$	$\mathrm{Sp}(n)$	2	1	conjugation by J
E_6	F_4	2	1	canonical involution
$\mathrm{Spin}(8)$	G_2	3	1	automorphism in [4]

Proposition 2.1. *Let G, H, p, d and ρ be as above. Then we have a homotopy equivalence*

$$\mathcal{G}_{dk}(G)_{\frac{1}{p}} \simeq \mathcal{G}_k(H)_{\frac{1}{p}} \times \Gamma(E)_{\frac{1}{p}},$$

where E is the pullback of $EH \times_\rho G/H$ by the map $S^4 \to BH$ representing $k \in \mathbf{Z} \cong \pi_4(BH)$.

Example 2.2. In [2], an involution of $\mathrm{Spin}(2n)$ whose fixed points subgroup is $\mathrm{Spin}(2n-1)$ is constructed. Harris [5] also showed that the localized map $\theta_{\frac{1}{2}}$ is a homotopy equivalence for this involution. Then we can apply Theorem 1.1 and Corollary 1.1. For this example, we can refine Proposition 2.1 a little. Put $n \geq 3$. Let E be the pullback of the bundle $E\,\mathrm{Spin}(2n-1) \times \rho S^{2n-1}$ by the map $S^4 \to B\,\mathrm{Spin}(2n-1)$ representing $k \in \mathbf{Z} \cong \pi_4(B\,\mathrm{Spin}(2n-1))$, where ρ is the restriction of the canonical action of $\mathrm{Spin}(2n)$ on S^{2n-1} to $\mathrm{Spin}(2n-1)$. Note that the composite

$$\pi_3(\mathrm{Spin}(2n-1)) \to \pi_3(\mathrm{Spin}(2n)) \to \pi_{2n+2}(S^{2n-1})$$

is the quotient map $\mathbf{Z} \to \mathbf{Z}/24$, where the first arrow is induced from the inclusion $\mathrm{Spin}(2n-1) \to \mathrm{Spin}(2n)$ and the second arrow is the J-homomorphism. Now we know that E is fibrewise homotopy equivalent to a fibre space E_k over S^4 with fibre S^{2n-1} classified by $[k] \in \mathbf{Z}/24 \cong \pi_{2n+2}(S^{2n-1})$. In particular, if k is a multiple of 3, $E^f_{\frac{1}{2}}$ is fibrewise homotopy equivalent to the trivial bundle $S^4 \times S^{2n-1}_{\frac{1}{2}}$. In this case, we have

$$\Gamma(E_k)_{\frac{1}{2}} \simeq \mathrm{map}(S^4, S^{2k-1}_{\frac{1}{2}}) \simeq S^{2n-1}_{\frac{1}{2}} \times \Omega^4 S^{2n-1}_{\frac{1}{2}},$$

since $S^{2n-1}_{\frac{1}{2}}$ is an H-space. Thus we have established:

Proposition 2.2 (cf. [11])**.** *Let E_k be as above. Then we have a homotopy equivalence*

$$\mathcal{G}_k(\mathrm{Spin}(2n))_{\frac{1}{2}} \simeq \mathcal{G}_k(\mathrm{Spin}(2n-1))_{\frac{1}{2}} \times \Gamma(E_k)_{\frac{1}{2}}.$$

Moreover, if k is a multiple of 3, *we have*

$$\Gamma(E_k)_{\frac{1}{2}} \simeq S^{2n-1}_{\frac{1}{2}} \times \Omega^4 S^{2n-1}_{\frac{1}{2}}.$$

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