On some estimates of the pressure for the Stokes equations in an infinite sector

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(Communicated by Masaki KASHIWARA, M.J.A., March 12, 2009)

Abstract: Some *a priori* estimates of the pressure for the Stokes equations in an infinite sector with the slip and the stress conditions on the boundary are established in weighted Sobolev spaces. The estimates for its higher order derivatives are obtained by the general scheme of Kondrat'ev. Instead the estimates for the lower order ones are derived by making use of the Mellin transformation and the explicit representation formula of the pressure.

Key words: Stokes equations; estimate of pressure; infinite sector; weighted Sobolev spaces.

1. Introduction. Let d_θ be a plane sector of opening θ, $\theta \in (0, 2\pi]$, in the polar coordinates (r, φ),

$$d_\theta = \{x = (r\cos\varphi, r\sin\varphi) \in \mathbf{R}^2 \mid r > 0, 0 < \varphi < \theta\}$$

with $\gamma_0 = \{\varphi = 0, r > 0\}$, $\gamma_\theta = \{\varphi = \theta, r > 0\}$ being the sides of the sector.

The following boundary value problem for the Stokes equations with a parameter $s \in \mathbf{C}$ is closely related to the evolution free boundary problem for the Navier-Stokes equations with contact lines (see [1]):

$$(1.1)\quad \begin{cases} s\mathbf{u} - \nu\Delta\mathbf{u} + \nabla q = \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0 \quad \text{in } d_\theta, \\ u_2|_{\gamma_0} = 0, \quad 2\nu D_{12}(\mathbf{u})|_{\gamma_0} = b_0, \\ \mathbf{P}(\mathbf{u}, q)\mathbf{n}_\theta|_{\gamma_\theta} = \mathbf{b}_\theta. \end{cases}$$

Here, $\mathbf{u} = \mathbf{u}(x, s) = (u_1(x, s), u_2(x, s))$ is the velocity vector field, $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$, $\mathbf{P}(\mathbf{u}, q) = -q\mathbf{I} + 2\nu\mathbf{D}(\mathbf{u})$ is the stress tensor, $\mathbf{D}(\mathbf{u})$ is the velocity deformation tensor with elements $D_{ij}(\mathbf{u}) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ $(i, j = 1, 2)$, $\mathbf{I}$ is the unit tensor of rank 2, $\mathbf{n}_\theta$ is the unit vector of the outward normal to γ_θ, $\mathbf{f} = (f_1, f_2)$, b_0, $\mathbf{b}_\theta$ are given functions on d_θ, γ_0 and γ_θ, respectively, and ν is a coefficient of viscosity, assumed to be a positive constant.

Similarly for the incompressible Navier-Stokes equations, in analyzing the problem (1.1) the estimates for the pressure q play an essential role. It is easy to derive the following problem for q from (1.1).

$$(1.2)\quad \begin{cases} \Delta q = \nabla\cdot\mathbf{f} \quad \text{in } d_\theta, \\ \left.\dfrac{\partial q}{\partial x_2}\right|_{\gamma_0} = f_2|_{\gamma_0} - \dfrac{\partial b_0}{\partial x_1}, \\ q|_{\gamma_\theta} = 2\nu\mathbf{D}(\mathbf{u})\mathbf{n}_\theta\cdot\mathbf{n}_\theta|_{\gamma_\theta} - \mathbf{b}_\theta\cdot\mathbf{n}_\theta. \end{cases}$$

In this paper we focus to the *a priori* estimates for q to the problem (1.2). Owing to the general theory of Kondrat′ev [3] such estimates can be derived. However his result leads to the estimates in higher order norms only, *i.e.*,

$$(1.3)\quad \|q\|^2_{\mathrm{H}^{2+k}_\mu(d_\theta)} = \sum_{|\alpha|\le 2+k}\int_{d_\theta} |x|^{2(\mu-2-k+|\alpha|)}|\mathrm{D}^\alpha_x q(x)|^2\,\mathrm{d}x \quad \text{for } k = 0, 1, 2, \ldots$$

The *a priori* estimates of q in lower order norms, especially of q itself, in the interior of a domain to problem (1.2) have its own interest, because in general it is essential for the solvability of the boundary value problems for partial differential equations of elliptic type even if the boundary is smooth.

Our main theorem is as follows:

Theorem 1. *Let $\theta \in (0, 2\pi]$ and $\mu \ne (m + 1/2)\pi/\theta$, $m \in \mathbf{Z}$. Suppose that $\mathbf{u}$, $\mathbf{f}$, b_0, and $\mathbf{b}_\theta$ are given functions satisfyng*

$$\|\nabla\mathbf{u}\|^2_{\mathbf{L}_{2,\mu-1/2}(\gamma_\theta)} \equiv \int_{\gamma_\theta} |\nabla\mathbf{u}|^2\, r^{2\mu-1}\,\mathrm{d}r < \infty,$$

2000 Mathematics Subject Classification. Primary 35Q30, 76D07, 76N10.

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doi: 10.3792/pjaa.85.37

$$\|\mathbf{f}\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} \equiv \int_{d_\theta} |\mathbf{f}|^2 |x|^{2\mu}\,\mathrm{d}x < \infty,$$

$$\|b_0\|^2_{\mathrm{L}_{2,\mu-1/2}(\gamma_0)} \equiv \int_{\gamma_0} |b_0|^2\, r^{2\mu-1}\,\mathrm{d}r < \infty,$$

$$\|\mathbf{b}_\theta\|^2_{\mathbf{L}_{2,\mu-1/2}(\gamma_\theta)} < \infty,$$

and q is a solution of (1.2). *Then we have the estimate*

$$\begin{aligned}(1.4)\quad \|q\|^2_{\mathrm{L}_{2,\mu-1}(d_\theta)} &\equiv \int_{d_\theta} |q|^2 |x|^{2(\mu-1)}\,\mathrm{d}x \\ &\leq \frac{c\theta}{\cos^2(\mu\theta)}\Big(\|\mathbf{f}\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} + \|b_0\|^2_{\mathrm{L}_{2,\mu-1/2}(\gamma_0)} \\ &\quad + \|\mathbf{b}_\theta\|^2_{\mathbf{L}_{2,\mu-1/2}(\gamma_\theta)} + \nu\|\nabla\mathbf{u}\|^2_{\mathbf{L}_{2,\mu-1/2}(\gamma_\theta)}\Big)\end{aligned}$$

with a constant c independent of θ, ν and μ.

We should be careful of the particular form of the right hand side and its dependence on the constants θ, ν and μ as in (1.4) since they are important for further studies about the wellposedness of various nonlinear problems with contact lines. Our method for it relies upon the explicit representation formula of q shown below (see also [4]).

Corollary 1. *Let θ, μ and $\mathbf{f}$ be as in Theorem* 1 *and ψ be the solution of problem*

$$(1.5)\qquad \begin{cases} \Delta\psi = \nabla\cdot\mathbf{f} \quad \text{in } d_\theta, \\ \dfrac{\partial\psi}{\partial x_2}\Big|_{\gamma_0} = f_2|_{\gamma_0}, \quad \psi|_{\gamma_\theta} = 0. \end{cases}$$

Then inequality

$$(1.6)\qquad \|\nabla\psi\|_{\mathbf{L}_{2,\mu}(d_\theta)} \leq c\|\mathbf{f}\|_{\mathbf{L}_{2,\mu}(d_\theta)}$$

holds for some constant c.

Remark. On the basis of estimates (1.4), (1.6) together with (1.3), after decomposing $\mathbf{f} \in \mathbf{L}_{2,\mu}(d_\theta)$ into $\mathbf{f} = \mathbf{f}^* + \nabla\psi$ with ψ being the solution of problem (1.5) and $\mathbf{f}^* = (f_1^*, f_2^*)$ satisfying $\nabla\cdot\mathbf{f}^* = 0$ in d_θ and $f_2^*|_{\gamma_0} = 0$ we shall establish in [2] the unique solvability of problem (1.1).

Proof of Corollary 1. We multiply equation (1.5) by $\psi|x|^{2\mu}$ and integrate over d_θ. After the integration by parts we obtain with the help of Young's inequality

$$\begin{aligned}&\|\nabla\psi\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} \\ &\quad\leq c\Big(\int_{d_\theta} |\mathbf{f}||\psi||x|^{2\mu-1}\,\mathrm{d}x + \int_{d_\theta} |\nabla\psi||\psi||x|^{2\mu-1}\,\mathrm{d}x \\ &\qquad + \int_{d_\theta} |\mathbf{f}||\nabla\psi||x|^{2\mu}\,\mathrm{d}x\Big) \\ &\quad\leq \varepsilon\|\nabla\psi\|^2_{\mathbf{L}_{2,\mu}(d_\theta)} + C_\varepsilon\Big(\|\psi\|^2_{\mathrm{L}_{2,\mu-1}(d_\theta)} + \|\mathbf{f}\|^2_{\mathbf{L}_{2,\mu}(d_\theta)}\Big)\end{aligned}$$

for any $\varepsilon > 0$. Then choosing ε suitably small and applying Theorem 1, one can deduce (1.6). ☐

2. Proof of Theorem 1. First of all we transform problem (1.2) by the polar coordinates (r, φ):

$$\begin{cases} \Big(\Big(r\dfrac{\partial}{\partial r}\Big)^2 + \Big(\dfrac{\partial}{\partial\varphi}\Big)^2\Big) q \\ \quad = r\dfrac{\partial}{\partial r}(rf_r) + \dfrac{\partial}{\partial\varphi}(rf_\varphi) \equiv g(r,\varphi) \quad \text{in } d_\theta, \\ \dfrac{\partial q}{\partial\varphi}\Big|_{\varphi=0} = rf_\varphi(r,0) - r\dfrac{\partial b_0}{\partial r} \equiv a_1(r), \\ q|_{\varphi=\theta} = 2\nu\mathbf{D}(\mathbf{u})\mathbf{n}_\theta\cdot\mathbf{n}_\theta|_{\gamma_\theta} - \mathbf{b}_\theta\cdot\mathbf{n}_\theta \equiv a_2(r), \end{cases}$$

where

$$\mathbf{f} = (f_1, f_2) = \big(f_r, f_\varphi\big)\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}.$$

By the Mellin transform with respect to r

$$\widetilde{b}(\lambda) = \int_0^\infty r^{\lambda-1} b(r)\,\mathrm{d}r, \quad \lambda = \lambda_1 + \mathrm{i}\lambda_2,$$

the following mixed problem of ordinary differential equations with respect to $\varphi \in (0,\theta)$ is derived:

$$(2.1)\qquad \begin{cases} \Big(\lambda^2 + \Big(\dfrac{\mathrm{d}}{\mathrm{d}\varphi}\Big)^2\Big)\widetilde{q}(\lambda,\varphi) \\ \quad = \widetilde{g}(\lambda,\varphi) \quad \text{on } (0,\theta), \\ \dfrac{\mathrm{d}\widetilde{q}}{\mathrm{d}\varphi}\Big|_{\varphi=0} = \widetilde{a}_1(\lambda), \quad \widetilde{q}|_{\varphi=\theta} = \widetilde{a}_2(r). \end{cases}$$

We seek a solution of problem (2.1) as $\widetilde{q} = \widetilde{q}_1 + \widetilde{q}_2$, where $\widetilde{q}_1$ and $\widetilde{q}_2$ are, respectively, solutions of

$$(2.2)\qquad \begin{cases} \Big(\lambda^2 + \Big(\dfrac{\mathrm{d}}{\mathrm{d}\varphi}\Big)^2\Big)\widetilde{q}_1(\lambda,\varphi) = 0 \quad \text{on } (0,\theta), \\ \dfrac{\mathrm{d}\widetilde{q}_1}{\mathrm{d}\varphi}\Big|_{\varphi=0} = \widetilde{a}_1(\lambda), \quad \widetilde{q}_1|_{\varphi=\theta} = \widetilde{a}_2(r) \end{cases}$$

and

$$(2.3)\qquad \begin{cases} \Big(\lambda^2 + \Big(\dfrac{\mathrm{d}}{\mathrm{d}\varphi}\Big)^2\Big)\widetilde{q}_2(\lambda,\varphi) = \widetilde{g}(\lambda,\varphi) \quad \text{on } (0,\theta), \\ \dfrac{\mathrm{d}\widetilde{q}_2}{\mathrm{d}\varphi}\Big|_{\varphi=0} = 0, \quad \widetilde{q}_2|_{\varphi=\theta} = 0. \end{cases}$$

It is not difficult to get the following explicit formulae for $\widetilde{q}_1$ and $\widetilde{q}_2$.

Lemma 1. *The solution of problem* (2.2) *is given by*

$$\widetilde{q}_1(\lambda,\varphi) = -\frac{\sin(\lambda(\theta-\varphi))}{\lambda\cos(\lambda\theta)}\,\widetilde{a}_1(\lambda) + \frac{\cos(\lambda\varphi)}{\cos(\lambda\theta)}\,\widetilde{a}_2(\lambda).$$

Here $\widetilde{a}_1(\lambda) = (\widetilde{rf_\varphi})(\lambda,0) + \lambda\tilde{b}_0(\lambda)$, so that

$$\begin{aligned}\widetilde{q}_1(\lambda,\varphi) &= -\frac{\sin(\lambda(\theta-\varphi))}{\lambda\cos(\lambda\theta)}\,(\widetilde{rf_\varphi})(\lambda,0)\\ &\quad -\frac{\sin(\lambda(\theta-\varphi))}{\cos(\lambda\theta)}\,\widetilde{b}_0(\lambda) + \frac{\cos(\lambda\varphi)}{\cos(\lambda\theta)}\,\widetilde{a}_2(\lambda).\end{aligned}$$

Lemma 2. *The solution of problem* (2.3) *is given by*

$$\begin{aligned}\widetilde{q}_2(\lambda,\varphi) = &-\int_0^\varphi \frac{\sin(\lambda(\theta-\varphi))\cos(\lambda\tau)}{\lambda\cos(\lambda\theta)}\,\widetilde{g}(\lambda,\tau)\,\mathrm{d}\tau\\ &-\int_\varphi^\theta \frac{\sin(\lambda(\theta-\tau))\cos(\lambda\varphi)}{\lambda\cos(\lambda\theta)}\,\widetilde{g}(\lambda,\tau)\,\mathrm{d}\tau.\end{aligned}$$

Since

$$\widetilde{g}(\lambda,\varphi) = -\lambda(\widetilde{rf_r}) + \frac{\mathrm{d}}{\mathrm{d}\varphi}(\widetilde{rf_\varphi}),$$

one can find by the integration by parts

$$\begin{aligned}&\widetilde{q}_2(\lambda,\varphi)\\ &= \int_0^\varphi \frac{\sin(\lambda(\theta-\varphi))\cos(\lambda\tau)}{\cos(\lambda\theta)}\,(\widetilde{rf_r})(\lambda,\tau)\,\mathrm{d}\tau\\ &\quad+\int_\varphi^\theta \frac{\sin(\lambda(\theta-\tau))\cos(\lambda\varphi)}{\cos(\lambda\theta)}\,(\widetilde{rf_r})(\lambda,\tau)\,\mathrm{d}\tau\\ &\quad-\int_0^\varphi \frac{\sin(\lambda(\theta-\varphi))\cos(\lambda\tau)}{\lambda\cos(\lambda\theta)}\,\frac{\mathrm{d}}{\mathrm{d}\tau}\,(\widetilde{rf_\varphi})(\lambda,\tau)\,\mathrm{d}\tau\\ &\quad-\int_\varphi^\theta \frac{\sin(\lambda(\theta-\tau))\cos(\lambda\varphi)}{\lambda\cos(\lambda\theta)}\,\frac{\mathrm{d}}{\mathrm{d}\tau}\,(\widetilde{rf_\varphi})(\lambda,\tau)\,\mathrm{d}\tau\\ &= \int_0^\varphi \frac{\sin(\lambda(\theta-\varphi))\cos(\lambda\tau)}{\cos(\lambda\theta)}\,(\widetilde{rf_r})(\lambda,\tau)\,\mathrm{d}\tau\\ &\quad+\int_\varphi^\theta \frac{\sin(\lambda(\theta-\tau))\cos(\lambda\varphi)}{\cos(\lambda\theta)}\,(\widetilde{rf_r})(\lambda,\tau)\,\mathrm{d}\tau\\ &\quad-\int_0^\varphi \frac{\sin(\lambda(\theta-\varphi))\sin(\lambda\tau)}{\cos(\lambda\theta)}\,(\widetilde{rf_\varphi})(\lambda,\tau)\,\mathrm{d}\tau\\ &\quad-\int_\varphi^\theta \frac{\cos(\lambda(\theta-\tau))\cos(\lambda\varphi)}{\cos(\lambda\theta)}\,(\widetilde{rf_\varphi})(\lambda,\tau)\,\mathrm{d}\tau\\ &\quad+\frac{\sin(\lambda(\theta-\varphi))}{\lambda\cos(\lambda\theta)}\,(\widetilde{rf_\varphi})(\lambda,0).\end{aligned}$$

Therefore the solution of (2.1) is expressed as

$$\begin{aligned}\text{(2.4)}\quad \widetilde{q}(\lambda,\varphi) &= \widetilde{q}_1(\lambda,\varphi) + \widetilde{q}_2(\lambda,\varphi)\\ &= \int_0^\varphi \frac{\sin(\lambda(\theta-\varphi))\cos(\lambda\tau)}{\cos(\lambda\theta)}\,(\widetilde{rf_r})(\lambda,\tau)\,\mathrm{d}\tau\\ &\quad+\int_\varphi^\theta \frac{\sin(\lambda(\theta-\tau))\cos(\lambda\varphi)}{\cos(\lambda\theta)}\,(\widetilde{rf_r})(\lambda,\tau)\,\mathrm{d}\tau\\ &\quad-\int_0^\varphi \frac{\sin(\lambda(\theta-\varphi))\sin(\lambda\tau)}{\cos(\lambda\theta)}\,(\widetilde{rf_\varphi})(\lambda,\tau)\,\mathrm{d}\tau\\ &\quad-\int_\varphi^\theta \frac{\cos(\lambda(\theta-\tau))\cos(\lambda\varphi)}{\cos(\lambda\theta)}\,(\widetilde{rf_\varphi})(\lambda,\tau)\,\mathrm{d}\tau\\ &\quad-\frac{\sin(\lambda(\theta-\varphi))}{\cos(\lambda\theta)}\,\widetilde{b}_0(\lambda) + \frac{\cos(\lambda\varphi)}{\cos(\lambda\theta)}\,\widetilde{a}_2(\lambda).\end{aligned}$$

By Parseval's equality

$$\int_0^\infty |b(r)|^2\, r^{2\lambda_1-1}\,\mathrm{d}r = \frac{1}{2\pi\mathrm{i}}\int_{\lambda_1-\mathrm{i}\infty}^{\lambda_1+\mathrm{i}\infty} |\widetilde{b}(\lambda)|^2\,\mathrm{d}\lambda,$$

we see

$$\begin{aligned}\|q\|^2_{\mathrm{L}_{2,\mu-1}(d_\theta)} &= \int_{d_\theta} |q|^2\,|x|^{2(\mu-1)}\,\mathrm{d}x\\ &= \int_0^\infty\int_0^\theta |q|^2\, r^{2\mu-1}\,\mathrm{d}\varphi\,\mathrm{d}r\\ &= \frac{1}{2\pi\mathrm{i}}\int_{\mu-\mathrm{i}\infty}^{\mu+\mathrm{i}\infty}\int_0^\theta |\widetilde{q}(\lambda,\varphi)|^2\,\mathrm{d}\varphi\,\mathrm{d}\lambda.\end{aligned}$$

Substitute (2.4) into the integrand and estimate each term. Last two terms of the right hand side of (2.4) can be estimated in the same way. Indeed, for the last term, since

$$\begin{aligned}\text{(2.5)}\quad &\left|\frac{\cos(\lambda\varphi)}{\cos(\lambda\theta)}\right|^2\\ &= \frac{\cos^2(\lambda_1\varphi)\cosh^2(\lambda_2\varphi)+\sin^2(\lambda_1\varphi)\sinh^2(\lambda_2\varphi)}{\cos^2(\lambda_1\theta)\cosh^2(\lambda_2\theta)+\sin^2(\lambda_1\theta)\sinh^2(\lambda_2\theta)}\\ &\le \frac{\cosh^2(\lambda_2\varphi)+\sinh^2(\lambda_2\varphi)}{\cos^2(\lambda_1\theta)\cosh^2(\lambda_2\theta)}\\ &\le \frac{\cosh^2(\lambda_2\theta)+\sinh^2(\lambda_2\theta)}{\cos^2(\lambda_1\theta)\cosh^2(\lambda_2\theta)}\\ &= \frac{1+\tanh^2(\lambda_2\theta)}{\cos^2(\lambda_1\theta)} \le \frac{2}{\cos^2(\lambda_1\theta)},\end{aligned}$$

we get

$$\begin{aligned}\text{(2.6)}\quad &\frac{1}{2\pi\mathrm{i}}\int_{\mu-\mathrm{i}\infty}^{\mu+\mathrm{i}\infty}\int_0^\theta \left|\frac{\cos(\lambda\varphi)}{\cos(\lambda\theta)}\right|^2 |\tilde{a}_2(\lambda)|^2\,\mathrm{d}\varphi\,\mathrm{d}\lambda\\ &\le \frac{1}{2\pi\mathrm{i}}\,\frac{2\theta}{\cos^2(\mu\theta)}\int_{\mu-\mathrm{i}\infty}^{\mu+\mathrm{i}\infty} |\tilde{a}_2(\lambda)|^2\,\mathrm{d}\lambda\end{aligned}$$

$$= \frac{2\theta}{\cos^2(\mu\theta)} \, \|a_2\|^2_{\mathrm{L}_{2,\mu-1/2}(\gamma_\theta)}.$$

Next we consider the other integral terms of the right hand side of (2.4). For the first term we have by virtue of the Schwarz inequality

$$\left| \int_0^\varphi \frac{\sin(\lambda(\theta-\varphi))\cos(\lambda\tau)}{\cos(\lambda\theta)} \, (\widetilde{rf_r})(\lambda,\tau)\,\mathrm{d}\tau \right|^2$$

$$\leq \left(\int_0^\varphi \left| \frac{\sin(\lambda(\theta-\varphi))\cos(\lambda\tau)}{\cos(\lambda\theta)} \right|^2 \mathrm{d}\tau \right)$$

$$\times \left(\int_0^\theta \left| (\widetilde{rf_r})(\lambda,\tau) \right|^2 \mathrm{d}\tau \right).$$

The integrand of the first integral in the right hand side is estimated as

$$(2.7) \left| \frac{\sin(\lambda(\theta-\varphi))\cos(\lambda\tau)}{\cos(\lambda\theta)} \right|^2$$

$$\leq \frac{1}{2} \, \frac{\left|\sin(\lambda(\theta-\varphi+\tau))\right|^2 + \left|\sin(\lambda(\theta-\varphi-\tau))\right|^2}{|\cos(\lambda\theta)|^2}.$$

Since $|\theta-\varphi\pm\tau| \leq \theta$ for $0 \leq \tau \leq \varphi \leq \theta$, we can estimate the right hand side of (2.7) in a similar way as (2.5). Hence

$$(2.8) \quad \frac{1}{2\pi\mathrm{i}} \int_0^\theta \int_{\mu-\mathrm{i}\infty}^{\mu+\mathrm{i}\infty} \left| \int_0^\varphi \frac{\sin(\lambda(\theta-\varphi))\cos(\lambda\tau)}{\cos(\lambda\theta)} \right.$$

$$\left. \times (\widetilde{rf_r})(\lambda,\tau)\,\mathrm{d}\tau \right|^2 \mathrm{d}\lambda\,\mathrm{d}\varphi$$

$$\leq \frac{1}{2\pi\mathrm{i}} \frac{4\theta}{\cos^2(\mu\theta)} \int_0^\theta \int_{\mu-\mathrm{i}\infty}^{\mu+\mathrm{i}\infty} \left| (\widetilde{rf_r})(\lambda,\varphi) \right|^2 \mathrm{d}\lambda\,\mathrm{d}\varphi$$

$$= \frac{1}{2\pi\mathrm{i}} \frac{4\theta}{\cos^2(\mu\theta)} \int_0^\theta \int_{\mu-\mathrm{i}\infty}^{\mu+\mathrm{i}\infty} \left| \widetilde{f}_r(\lambda+1,\varphi) \right|^2 \mathrm{d}\lambda\,\mathrm{d}\varphi$$

$$= \frac{1}{2\pi\mathrm{i}} \frac{4\theta}{\cos^2(\mu\theta)} \int_0^\theta \int_{\mu+1-\mathrm{i}\infty}^{\mu+1+\mathrm{i}\infty} \left| \widetilde{f}_r(\lambda,\varphi) \right|^2 \mathrm{d}\lambda\,\mathrm{d}\varphi$$

$$= \frac{4\theta}{\cos^2(\mu\theta)} \int_0^\theta \int_0^\infty |f_r(r,\varphi)|^2 \, r^{2(\mu+1)-1} \,\mathrm{d}r\,\mathrm{d}\varphi$$

$$\leq \frac{4\theta}{\cos^2(\mu\theta)} \, \|\mathbf{f}\|^2_{\mathbf{L}_{2,\mu}(d_\theta)}.$$

From (2.6) and (2.8) we can deduce the desired estimate.

Almost the same calculations as above bring the similar estimates for other integral terms, and finally (1.4) is concluded.

Thus, the Proof of Theorem 1 is completed.

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