On the analysis of the scattering problem for the elastic wave in the case of the transverse incident wave

By Yasushi OTA

Department of Mathematics, Osaka University, 1-1, Machikaneyama, Toyonaka, Osaka 560-0043, Japan

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Abstract: When we analyze the reflection phenomenon for the elastic wave, the one of the most complicated and interesting problems is to study the mode conversion case. For the elastic wave, there are waves of different modes and a remarkable phenomenon called "mode-conversion" which causes serious difficulties. In this paper, by considering the non back-scattering case, we examine the singularities of the scattering kernel for the elastic wave equation with transverse incident waves and derive a new result about the singularities of the scattering kernel.

Key words: Scattering; elastic equation; mode-conversion.

1. Introduction. Let Ω be an exterior domain in $\mathbf{R}^3$ with smooth and compact boundary. We consider the isotropic elastic wave equation with the Dirichlet boundary condition

(1.1)
$$\begin{cases} (\partial_t^2 - L)u(t,x) = 0 & \text{in } \mathbf{R}\times\Omega, \\ u(t,x) = 0 & \text{on } \mathbf{R}\times\partial\Omega, \\ u(0,x) = f_1(x) \quad \partial_t u(0,x) = f_2(x) & \text{on } \Omega, \end{cases}$$

where $u(t,x) = {}^t(u_1,u_2,u_3)$ and $f_i(x) = {}^t(f_{i1}, f_{i2}, f_{i3})$ $(i = 1,2)$. Recall that L has the following form:

$$L = \sum_{i,j=1}^{3} a_{ij}\partial_{x_i}\partial_{x_j},$$

where a_{ij} are 3×3 matrices of which (p,q)-entry is expressed by a_{ipjq}. We say that the elastic medium Ω is isotropic, if a_{ipjq} is given by

$$a_{ipjq} = \lambda\delta_{ip}\delta_{jq} + \mu(\delta_{ij}\delta_{pq} + \delta_{iq}\delta_{jp}),$$

where λ, μ are Lame's constants satisfying the following inequalities:

$$\lambda + \frac{2}{3}\mu > 0, \quad \mu > 0.$$

Under the assumption that the elastic medium Ω is isotropic, Yamamoto [13] and Shibata-Soga [8] have formulated a scattering theory which is analogous to the theory of Lax-Phillips [4]. Let $k_-(s,\omega)$ and $k_+(s,\omega) \in L^2(\mathbf{R}\times S^2)$ denote the incoming and outgoing translation representations of an initial data $f = {}^t(f_1, f_2)$ respectively (see [4]). Recall that the scattering operator S is the mapping

$$S: \ k_-(s,\omega) \ \longmapsto \ k_+(s,\omega).$$

The scattering operator S admits a representation:

$$(Sk_-)(s,\theta) = \iint_{\mathbf{R}\times S^2} S(s-\tilde{s},\theta,\omega)k_-(\tilde{s},\omega)d\tilde{s}d\omega$$

with a distribution kernel $S(s,\theta,\omega)$ called the scattering kernel. Majda [5] has obtained a representation formula of the scattering kernel $S(s,\theta,\omega)$ for the scalar-valued case. This representation formula is very effective to investigate inverse scattering problems (cf. Majda [5], Soga [9], Petkov [7]). For the elastic case, Soga [10] and Kawashita [2] have derived a representation formula of the scattering kernel.

The characteristic matrix $L(\xi)$ of the operator $L(\partial_x)$ has the eigenvalues $C_1^2|\xi|^2$ and $C_2^2|\xi|^2$, where

$$C_1 = (\lambda + 2\mu)^{\frac{1}{2}}, \quad C_2 = \mu^{\frac{1}{2}}.$$

Let $P_i(\xi)$ be the eigenprojector associated to the eigenvalues $C_i^2|\xi|^2 (i = 1,2)$, where

$$P_1(\xi) = \xi\otimes\xi, \quad P_2(\xi) = I - P_1(\xi).$$

Then $P_1(\xi)\mathbf{R}^3$ is the space spaned by ξ, and $P_2(\xi)\mathbf{R}^3$ is the orthogonal complement of $P_1(\xi)\mathbf{R}^3$. Associated with the eigenvalues $C_i^2|\xi|^2 (i = 1,2)$, there are waves of two different types (modes). The one propagates with the speed C_1, and the other with C_2. Furthermore their amplitudes are longitudinal and transverse to the propagation direction respectively, and therefore these waves are called longitudinal and transverse waves respectively. For elastic

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waves there is a remarkable phenomenon called "mode-conversion", that is, when longitudinal or transverse incident wave hits the boundary $\partial\Omega$, both longitudinal reflected wave and transverse reflected wave appear. This phenomenon causes serious difficulties in the analysis of singularities of the scattering kernel for the elastic wave equation.

In view of results concerning mode-conversion (cf. Chapter 5 of Achenbach [1] and Theorem 2.1 of Soga [12]), we can expect that corresponding phenomenon occurs for the scattering kernel $S(s,\theta,\omega)$, because in the asymptotic sense the kernel $P_i(\theta)S(C_i^{-\frac{1}{2}}\theta\cdot x - t,\theta,\omega)P_l(\omega)$ expresses the C_i-mode component of the scattered wave in the direction θ for the C_l-mode incident plane wave in the direction ω. In the back-scattering case (i.e $\theta=-\omega$), studying by Soga [10, 11] we can obtain the same results as in Majda [5]. In [11], he has derived an asymptotic expansion of $P_i(-\omega)S(\cdot,-\omega,\omega)P_l(\omega)$ which is valid near the right end point of the singular support for $s\in\mathbf{R}$ (i.e $s=-r_{il}(\omega)$):

(1.2)

$$P_i(-\omega)S(s,-\omega,\omega)P_l(\omega) \sim c\sum_{k=1}^{N} K(a_k)^{-\frac{1}{2}}\delta^{(1)}(-s-r_{il}(\omega))P_i(\omega)P_l(\omega)+\cdots,$$

where $r_{il}(\omega):=(C_i^{-1}+C_l^{-1})r(\omega), r(\omega):=\min_{s\in\partial\Omega} x\cdot\omega$, $\{x;\ \omega\cdot x=r(\omega)\}\cap\partial\Omega=\{a_t\}_{t=1,\cdots,N}$ and $K(a_t)$ is the Gaussian curvature of $\partial\Omega$ at a_t and c is a constant. For a distribution $f(s)$ on $\mathbf{R}$ we use the notation

$$f(s)\sim f_0(s)+f_1(s)+\cdots \quad \text{at } s_0,$$

which means that there exists an integer m and a C^∞ function $\varphi(s)$ with $\varphi(s_0)\neq 0$ such that for every integer $N\geq 0$

$$\varphi(s)\{f(s)-(f_0(s)+\cdots+f_N(s))\}\in H^{m+N}(\mathbf{R}).$$

For the detailed proof, see Theorem 6.1 in [11]. Since $P_i(\omega)P_l(\omega)=0$ in the mode-conversion case (i.e $i\neq l$), in the analysis of the singularity we can use the above asymptotic expansion (1.2) only when $i=l$. In the mode-conversion case, studying by Kawashita-Soga [3], it is necessary to examine the lower term of the asymptotic expansion of the scattering kernel. In [6], we have proved the following results:

(i) supp $[P_i(\theta)S(\cdot,\theta,\omega)P_1(\theta)]\subset(-\infty,-r_{i1}(\theta,\omega)]$ $(i=1,2)$,

(ii) $P_i(\theta)S(s,\theta,\omega)P_1(\omega)$ is singular (not C^∞) at $s=-r_{i1}(\theta,\omega)(i=1,2)$,

where $r_{il}(\theta,\omega) := \min_{x\in\partial\Omega} x\cdot n_{il}(\theta,\omega), n_{il}(\theta,\omega) := -(C_i^{-1}\theta - C_l^{-1}\omega)$.

In this paper we examine the singularities of the scattering kernel in the case of transverse incident wave, which was not treated in Ota [6].

Before giving the main results in the present paper, we give several definitions for stating those. We denote the first hitting points at $\partial\Omega$ by $N_{il}(\theta,\omega) := \{x; n_{il}(\theta,\omega)\cdot x = r_{il}(\theta,\omega)\}\cap\partial\Omega$. Furthermore, we arbitrarily pick a point $a_t\in N_{il}(\theta,\omega)$ and choose a system of orthogonal local coordinates $y=(y',y_3)$, with $y'=(y_1,y_2)$, in $\mathbf{R}^3$ such that $y_3=(r_{il}(\theta,\omega) - n_{il}(\theta,\omega)\cdot x)|n_{il}(\theta,\omega)|^{-1}$, and that $y=0$ expresses the reference point a_t. Then Ω is represented by $y_3>\psi(y')$ in a neighborhood U of a_t, where $\psi(y')$ is a C^∞ function defined in a neighborhood of $y'=0$.

If the Hessian matrix $H_{\psi(y')}$ of $\psi(y')$ is negative definite at $y'=0$ for every such picked point, we say that $n_{ij}(\theta,\omega)$ is a regular direction for $\partial\Omega$, which does not depend on the choice of the coordinates $y=(y',y_3)$. If $n_{il}(\theta,\omega)$ is a regular direction, the set $N_{il}(\theta,\omega)$ consists of a finite number of isolated points.

From the above definitions we give the following result.

Theorem 1.1. *Let $\omega,\theta\in S^2$. Assume that $|\theta+\omega|$ is different from zero and sufficiently small, and $n_{i2}(\theta,\omega)$ is a regular direction for $\partial\Omega$. Then we have*

$P_i(\theta)S(s,\theta,\omega)P_2(\omega)$ is singular (not C^∞) at $s=-r_{i2}(\theta,\omega)\quad(i=1,2)$.

The basic idea of the proof is to derive an asymptotic expansion of $P_i(\theta)S(s,\theta,\omega)P_2(\omega)$ and to show that the leading term of this asymptotic expansion don't vanish. In this paper we examine only the singularity of $P_i(\theta)S(s,\theta,\omega)P_2(\omega)$ (i.e the case of (ii) of Theorem 2.1 in [6]), because we are not able to apply the method of the finite propagation speed which was used in [6] to study the support of $P_i(\theta)S(s,\theta,\omega)P_2(\omega)$.

2. Asymptotic expansion of the scattering kernel and Proof of Theorem 1.1. In order to prove the singularities of $P_i(\theta)S(s,\theta,\omega)P_2(\omega)$ we make use of an asymptotic expansion of the scattering kernel which is derived from Proposition 3.3 in [6] and prove Theorem 1.1.

To obtain Proposition 3.3 in the case of $l=2$ in [6], we take an orthonormal frame $\{p_1, p_2, p_3\}$ where $p_3 = -n_{il}(\theta,\omega)|n_{il}(\theta,\omega)|^{-1}$, and choose the local coordinate system $y=(y_1,y_2,y_3)$ such that $x = y_1p_1 + y_2p_2 + y_3p_3$. Moreover we denoted by T the 3×3 orthogonal matrix $T=(t_{pq})$ such that $T(e_j) = p_j (j=1,2,3)$, where $\{e_1,e_2,e_3\}$ is the canonical basis in $\mathbf{R}^3$. Then we can obtain the following asymptotic expansion by using Proposition 3.3 in the case of $l=2$.

Proposition 2.1. *Let $\omega,\theta \in S^2$. Assume that $|\theta+\omega|$ is sufficiently small, and $n_{i2}(\theta,\omega)$ is a regular direction for $\partial\Omega$. Then we have*

$$\begin{aligned}
&P_i(\theta)S(s,\theta,\omega)P_2(\omega)\\
&\sim (2\sqrt{2}\pi)^{-2} C_i^{-\frac{3}{2}} C_2^{-\frac{3}{2}} |n_{i2}(\tilde\theta,\tilde\omega)|^{-2}\delta^{(1)}(-s - r_{i2}(\tilde\theta,\tilde\omega))\\
&\times \sum_{t=1}^{M} K(a_t)^{-\frac{1}{2}} |S^1| T\Bigg[P_i(\tilde\theta)\sum_{k=1}^{2}\Bigg\{\sum_{q=1}^{2}\Big\{a_{3q}(C_2^{-1}\tilde\omega_q)\\
&+ {}^t a_{3q}(C_i^{-1}\tilde\theta_q)\Big\} + a_{33}(C_2^{-1}\tilde\kappa_{k2} + C_i^{-1}\tilde\theta_3)\Bigg\}\tilde P^2_{k,0}P_2(\tilde\omega)\Bigg]{}^tT\\
&+\cdots,
\end{aligned}$$

where $x = Ty$, $\tilde\omega = {}^tT\omega$, $\tilde\theta = {}^tT\theta$ and $\tilde\kappa_{k2} = \sqrt{\tilde\omega_3^2 + C_2^2\cdot C_k^{-2} - 1}$. Moreover $\tilde P^2_{1,0} = \frac{1}{\tilde p\cdot\tilde q}(\tilde p\otimes\tilde q)$ and $\tilde P^2_{2,0} = \frac{1}{\tilde p\cdot\tilde q}\{(\tilde p\cdot\tilde q)I - \tilde p\otimes\tilde q\}$ where $\tilde p = {}^t(\tilde\omega_1,\tilde\omega_2,\tilde\kappa_{12})$, $\tilde q = {}^t(\tilde\omega_1,\tilde\omega_2,|\tilde\omega_3|)$.

For the detailed proof, see Propositon 3.3 in [6].

Next, using Proposition 2.1, we investigate the singularities of the scattering kernel $S(s,\theta,\omega)$ for the non-back scattering case and prove Theorem 1.1.

Proof of Theorem 1.1. Note that $P_1(\xi) = \xi\otimes\xi$, $P_2(\xi) = I - \xi\otimes\xi$ and each $\tilde P^2_{k,0}P_2(\tilde\omega) (k=1,2)$ takes the following form:

$$\begin{aligned}
(2.1)\quad \bar P_1(\tilde\omega) &:= \tilde P_1, 0^2 P_2(\tilde\omega)\\
&= \frac{2}{\tilde p\cdot\tilde q}\begin{pmatrix} \tilde\omega_1^2\tilde\omega_3^2 & \tilde\omega_1\tilde\omega_2\tilde\omega_3^2 & \tilde\omega_1|\tilde\omega_3|(1-\tilde\omega_3^2)\\ \tilde\omega_2\tilde\omega_1\tilde\omega_3^2 & \tilde\omega_2^2\tilde\omega_3^2 & \tilde\omega_2|\tilde\omega_3|(1-\tilde\omega_3^2)\\ \tilde\kappa_{12}\tilde\omega_1\tilde\omega_3^2 & \tilde\kappa_{12}\tilde\omega_2\tilde\omega_3^2 & \tilde\kappa_{12}|\tilde\omega_3|(1-\tilde\omega_3^2)\end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
(2.2)\quad &\bar P_2(\tilde\omega) := \tilde P^2_{2,0}P_2(\tilde\omega) = \frac{1}{\tilde p\cdot\tilde q}\\
&\times\begin{pmatrix} \tilde p\cdot\tilde q - \tilde\omega_1^2 z_+ & -\tilde\omega_1\tilde\omega_2 z_+ & \tilde\omega_1|\tilde\omega_3|z_-\\ -\tilde\omega_2\tilde\omega_1 z_+ & \tilde p\cdot\tilde q - \tilde\omega_2^2 z_+ & \tilde\omega_2|\tilde\omega_3|z_-\\ -\tilde\omega_3\tilde\omega_1(\tilde\omega\cdot\tilde p) & -\tilde\omega_3\tilde\omega_2(\tilde\omega\cdot\tilde p) & (1-\tilde\omega_3^2)(\tilde\omega\cdot\tilde p)\end{pmatrix},
\end{aligned}$$

where $z_\pm := |\tilde\omega_3|(\tilde\kappa_{12} - \tilde\omega_3) \pm 1$.

Note that $\tilde\omega_3 < 0$ and recalling that $n_{i2}(\tilde\theta,\tilde\omega)/|n_{i2}(\tilde\theta,\tilde\omega)| = {}^t(0,0,-1)$, we can rewrite $P_i(\tilde\theta)\sum_{k=1}^{2}\Big[\sum_{q=1}^{2}\{a_{3q}(C_2^{-1}\tilde\omega_q) + {}^t a_{3q}(C_i^{-1}\tilde\theta_q)\} + a_{33}(C_2^{-1}\tilde\kappa_{k2} + C_i^{-1}\tilde\theta_3)\Big]\tilde P^2_{k,0}P_2(\tilde\omega)$ in the following form:

$$\begin{aligned}
(2.3)\quad P_i(\tilde\theta)\sum_{k=1}^{2}\big\{&(a_{31} + {}^ta_{31})\,\tilde\omega_1 + (a_{32} + {}^ta_{32})\tilde\omega_2\\
&+ a_{33}(\tilde\omega_3 + \tilde\kappa_{k2} + C_2|\tilde n_{i2}|)\big\}\bar P_k(\tilde\omega)/C_2.
\end{aligned}$$

Then, calculating each term in (2.3) more carefully, we can obtain

$$\begin{aligned}
&P_i(\tilde\theta)\sum_{k=1}^{2}\begin{pmatrix}0 & 0 & \lambda+\mu\\ 0 & 0 & 0\\ \lambda+\mu & 0 & 0\end{pmatrix}\tilde\omega_1\bar P_k(\tilde\omega)\\
&\quad = (\lambda+\mu)\tilde\omega_1\big\{2(a\otimes d) + \tilde p_{i3}\otimes\bar a + \tilde p_{i1}\otimes\bar c\big\}/(\tilde p\cdot\tilde q),
\end{aligned}$$

$$\begin{aligned}
&P_i(\tilde\theta)\sum_{k=1}^{2}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & \lambda+\mu\\ 0 & \lambda+\mu & 0\end{pmatrix}\tilde\omega_2\bar P_k(\tilde\omega)\\
&\quad = (\lambda+\mu)\tilde\omega_2\big\{2(b\otimes d) + \tilde p_{i3}\otimes\bar b + \tilde p_{i2}\otimes\bar c\big\}/(\tilde p\cdot\tilde q),
\end{aligned}$$

$$\begin{aligned}
&P_i(\tilde\theta)\sum_{k=1}^{2}\begin{pmatrix}\mu & 0 & 0\\ 0 & \mu & 0\\ 0 & 0 & \lambda+2\mu\end{pmatrix}(\tilde\omega_3 + \tilde\kappa_{k2} + C_2|\tilde n_{i2}|)\bar P_k(\tilde\omega)\\
&= \Big[2(\tilde\omega_3 + \tilde\kappa_{12} + C_2|\tilde n_{i2}|)(c\otimes d)\\
&\quad + C_2|\tilde n_{i2}|\big\{\mu(\tilde p_{i1}\otimes\bar a + \tilde p_{i2}\otimes\bar b)\\
&\quad + (\lambda+2\mu)\tilde p_{i3}\otimes\bar c\big\}\Big]/(\tilde p\cdot\tilde q),
\end{aligned}$$

where

$$\begin{aligned}
a &= {}^t(\tilde\omega_1\tilde p_i^{(13)} + \tilde\kappa_{12}\tilde p_i^{(11)}, \tilde\omega_1\tilde p_i^{(23)} + \tilde\kappa_{12}\tilde p_i^{(21)},\\
&\qquad \tilde\omega_1\tilde p_i^{(33)} + \tilde\kappa_{12}\tilde p_i^{(31)}),\\
\bar a &= {}^t(\tilde p\cdot\tilde q - \tilde\omega_1^2 z_+, -\tilde\omega_1\tilde\omega_2 z_+, \tilde\omega_1|\tilde\omega_3|z_-)\\
b &= {}^t(\tilde\omega_2\tilde p_i^{(13)} + \tilde\kappa_{12}\tilde p_i^{(12)}, \tilde\omega_2\tilde p_i^{(23)} + \tilde\kappa_{12}\tilde p_i^{(22)},\\
&\qquad \tilde\omega_2\tilde p_i^{(33)} + \tilde\kappa_{12}\tilde p_i^{(32)}),\\
\bar b &= {}^t(-\tilde\omega_2\tilde\omega_1 z_+, \tilde p\cdot\tilde q - \tilde\omega_2^2 z_+, \tilde\omega_2|\tilde\omega_3|z_-)\\
c &= {}^t(\mu(\tilde\omega_1\tilde p_i^{(11)} + \tilde\omega_2\tilde p_i^{(12)}) + (\lambda+2\mu)\tilde\kappa_{12}\tilde p_i^{(13)},\\
&\qquad \mu(\tilde\omega_1\tilde p_i^{(21)} + \tilde\omega_2\tilde p_i^{(22)}) + (\lambda+2\mu)\tilde\kappa_{12}\tilde p_i^{(23)},\\
&\qquad \mu(\tilde\omega_1\tilde p_i^{(31)} + \tilde\omega_2\tilde p_i^{(32)}) + (\lambda+2\mu)\tilde\kappa_{12}\tilde p_i^{(33)}),\\
\bar c &= {}^t(-\tilde\omega_3\tilde\omega_1(\tilde\omega\cdot\tilde p), -\tilde\omega_3\tilde\omega_2(\tilde\omega\cdot\tilde p), (1-\tilde\omega_3^2)(\tilde\omega\cdot\tilde p)),\\
d &= {}^t(\tilde\omega_1\tilde\omega_3^2, \tilde\omega_2\tilde\omega_3^2, |\tilde\omega_3|(1-\tilde\omega_3^2)).
\end{aligned}$$

each $\tilde{p}_i^{(pq)}$ and $\tilde{p}_{il}$ denote (p,q)-entry and l-th column of $P_i(\tilde{\theta})$ respectively, and $\tilde{n}_{i2}=n_{i2}(\tilde{\theta},\tilde{\omega})$. Hence, applying the asymptotic expansion derived in the Proposition 2.1, we can obtain

(2.4)
$$\begin{aligned}P_i(\theta)S(s,\theta,\omega)P_2(\omega)\\ &\sim (2\sqrt{2}\pi)^{-2}C_2^{-\frac{5}{2}}C_i^{-\frac{3}{2}}\delta^{(1)}(-s-r_{i1}(\tilde{\theta},\tilde{\omega}))\\ &\quad\times\sum_{t=1}^{M}K(a_t)^{-\frac{1}{2}}|S^1|TM(\tilde{\theta},\tilde{\omega})^tT+\cdots,\end{aligned}$$

where $M(\tilde{\theta},\tilde{\omega})$ is a 3×3-matrix whose (p,q)-entry is expressed by $m_{pq}(\tilde{\theta},\tilde{\omega})$. As is shown above, it is represented in the following form:

(2.5)
$$\begin{aligned}&M(\tilde{\theta},\tilde{\omega})\\ &=\Big[(\lambda+\mu)\tilde{\omega}_1\big\{2(a\otimes d)+\tilde{p}_{i3}\otimes\bar{a}+\tilde{p}_{i1}\otimes\bar{c}\big\}\\ &\quad+(\lambda+\mu)\tilde{\omega}_2\big\{2(b\otimes d)+\tilde{p}_{i3}\otimes\bar{b}+\tilde{p}_{i2}\otimes\bar{c}\big\}\\ &\quad+2(\tilde{\omega}_3+\tilde{\kappa}_{12}+C_2|\tilde{n}_{i2}|)(c\otimes d)\\ &\quad+C_2|\tilde{n}_{i2}|\big\{\mu(\tilde{p}_{i1}\otimes\bar{a}+\tilde{p}_{i2}\otimes\bar{b})\\ &\quad+(\lambda+2\mu)\tilde{p}_{i3}\otimes\bar{c}\big\}\Big]/(\tilde{p}\cdot\tilde{q}).\end{aligned}$$

Finally, by considering the mode conversion case and non-mode conversion case separately, we show that the leading term of the right hand side of (3.4) does not vanish. To show it, we first observe the mode-conversion case ($i=1$).

Lemma 2.2. *Assume that $|\tilde{\theta}+\tilde{\omega}|$ is different from zero and sufficiently small. Then we have $m_{33}(\tilde{\theta},\tilde{\omega})\neq 0$.*

Proof. Let $i=1$. According to (2.5), $m_{33}(\tilde{\theta},\tilde{\omega})$ is expressed as follows:

$$\begin{aligned}&m_{33}(\tilde{\theta},\tilde{\omega})\\ &=\Big[(\lambda+\mu)\tilde{\omega}_1\big\{2(\tilde{\omega}_1\tilde{p}_1^{(33)}+\tilde{\kappa}_{12}\tilde{p}_1^{(31)})|\tilde{\omega}_3|(1-\tilde{\omega}_3^2)\\ &\quad+\tilde{p}_1^{(33)}\tilde{\omega}_1|\tilde{\omega}_3|z_-+\tilde{p}_1^{(31)}(1-\tilde{\omega}_3^2)(\tilde{\omega}\cdot\tilde{p})\big\}\\ &\quad+(\lambda+\mu)\tilde{\omega}_2\big\{2(\tilde{\omega}_2\tilde{p}_1^{(33)}+\tilde{\kappa}_{12}\tilde{p}_1^{(32)})|\tilde{\omega}_3|(1-\tilde{\omega}_3^2)\\ &\quad+\tilde{p}_1^{(33)}\tilde{\omega}_2|\tilde{\omega}_3|z_-+\tilde{p}_1^{(32)}(1-\tilde{\omega}_3^2)(\tilde{\omega}\cdot\tilde{p})\big\}\\ &\quad+2(\tilde{\omega}_3+\tilde{\kappa}_{12}+C_2|\tilde{n}_{12}|)\big\{\mu(\tilde{\omega}_1\tilde{p}_1^{(31)}+\tilde{\omega}_2\tilde{p}_1^{(32)})\\ &\quad+(\lambda+2\mu)\tilde{\kappa}_{12}\tilde{p}_1^{(33)}\big\}|\tilde{\omega}_3|(1-\tilde{\omega}_3^2)\\ &\quad+C_2|\tilde{n}_{12}|\big\{\mu|\tilde{\omega}_3|z_-(\tilde{\omega}_1\tilde{p}_1^{(31)}+\tilde{\omega}_2\tilde{p}_1^{(32)})\\ &\quad+(\lambda+2\mu)\tilde{p}_1^{(33)}(1-\tilde{\omega}_3^2)(\tilde{\omega}\cdot\tilde{p})\big\}\Big]/(\tilde{p}\cdot\tilde{q})\\ &=\Big[(\lambda+\mu)|\tilde{\omega}_3|(1-\tilde{\omega}_3^2)\big\{2(1-\tilde{\omega}_3^2)+z_-\big\}\tilde{p}_1^{(33)}\\ &\quad+(\lambda+\mu)(1-\tilde{\omega}_3^2)\big\{2\tilde{\kappa}_1 2|\tilde{\omega}_3|+(\tilde{\omega}\cdot\tilde{p})\big\}\\ &\quad\times(\tilde{\omega}_1\tilde{p}_1^{(31)}+\tilde{\omega}_2\tilde{p}_1^{(32)})\\ &\quad+\mu|\tilde{\omega}_3|\big\{2(\tilde{\omega}_3+\tilde{\kappa}_{12}+C_2|\tilde{n}_{12}|)(1-\tilde{\omega}_3^2)\\ &\quad+C_2|\tilde{n}_{12}|z_-\big\}(\tilde{\omega}_1\tilde{p}_1^{(31)}+\tilde{\omega}_2\tilde{p}_1^{(32)})\\ &\quad+(\lambda+2\mu)(1-\tilde{\omega}_3^2)\big\{2(\tilde{\omega}_3+\tilde{\kappa}_{12}+C_2|\tilde{n}_{12}|)\tilde{\kappa}_{12}|\tilde{\omega}_3|\\ &\quad+C_2|\tilde{n}_{12}|(\tilde{\omega}\cdot\tilde{p})\big\}\tilde{p}_1^{(33)}\Big]/(\tilde{p}\cdot\tilde{q}).\end{aligned}$$

By $\tilde{n}_{12}(\tilde{\theta},\tilde{\omega})/|\tilde{n}_{12}(\tilde{\theta},\tilde{\omega})|=(0,0,-1)$, that is,

$$C_2^{-1}\tilde{\omega}_p=C_1^{-1}\tilde{\theta}_p\ (p=1,2),$$
$$C_2^{-1}\tilde{\omega}_3=C_1^{-1}\tilde{\theta}_3-|\tilde{n}_{12}(\tilde{\theta},\tilde{\omega})|\text{ and }|\theta|=1,$$

we can express $m_{33}(\tilde{\theta},\tilde{\omega})$ as a function in $(\tilde{\theta}_1,\tilde{\theta}_2)$:

$$m_{33}(\tilde{\theta}_1,\tilde{\theta}_2)=F(\tilde{\theta}_1,\tilde{\theta}_2)/(\tilde{p}\cdot\tilde{q}),$$

where

$$\begin{aligned}&F(\tilde{\theta}_1,\tilde{\theta}_2)\\ &=(\lambda+\mu)|\tilde{\omega}_3(\tilde{\theta})|C_1^{-2}C_2^2(\tilde{\theta}_1^2+\tilde{\theta}_2^2)\\ &\quad\times\big\{2C_1^{-2}C_2^2(\tilde{\theta}_1^2+\tilde{\theta}_2^2)+z_-\big\}\{1-(\tilde{\theta}_1^2+\tilde{\theta}_2^2)\}\\ &\quad+(\lambda+\mu)C_1^{-2}C_2^2(\tilde{\theta}_1^2+\tilde{\theta}_2^2)\\ &\quad\times\big\{2\tilde{\kappa}_{12}|\tilde{\omega}_3(\tilde{\theta})|+\tilde{\omega}(\theta)\cdot\tilde{p}\big\}C_1^{-1}C_2\tilde{\theta}_3(\tilde{\theta})(\tilde{\theta}_1^2+\tilde{\theta}_2^2)\\ &\quad+\mu C_1^{-2}C_2^2(\tilde{\theta}_1^2+\tilde{\theta}_2^2)|\tilde{\omega}_3(\tilde{\theta})|\\ &\quad\times\big\{2(\tilde{\omega}_3(\tilde{\theta})+\tilde{\kappa}_{12}+C_2|\tilde{n}_{12}|)C_1^{-2}C_2^2(\tilde{\theta}_1^2+\tilde{\theta}_2^2)\\ &\quad+C_2|\tilde{n}_{12}|z_-\big\}C_1C_2^{-1}\tilde{\theta}_3(\tilde{\theta})\\ &\quad+(\lambda+2\mu)C_1^{-2}C_2^2(\tilde{\theta}_1^2+\tilde{\theta}_2^2)\\ &\quad\times\big\{2(\tilde{\omega}_3(\tilde{\theta})+\tilde{\kappa}_{12}+C_2|\tilde{n}_{12}|)\tilde{\kappa}_{12}|\tilde{\omega}_3(\tilde{\theta})|\\ &\quad+C_2|\tilde{n}_{12}|(\tilde{\omega}(\theta)\cdot\tilde{p})\big\}\tilde{\theta}_3^2(\tilde{\theta})\\ &=C_1^{-2}C_2^2(\tilde{\theta}_1^2+\tilde{\theta}_2^2)\tilde{F}(\tilde{\theta}_1,\tilde{\theta}_2).\end{aligned}$$

Here we note that $|\tilde{\theta}+\tilde{\omega}|\neq 0$ is equivalent to $(\tilde{\theta}_1,\tilde{\theta}_2)\neq(0,0)$. To prove $m_{33}(\tilde{\theta},\tilde{\omega})\neq 0$, it suffices to show that $\tilde{F}(\tilde{\theta}_1,\tilde{\theta}_2)\neq 0$. Since $\tilde{F}(\tilde{\theta}_1,\tilde{\theta}_2)$ is a C^∞ function near $(\tilde{\theta}_1,\tilde{\theta}_2)=(0,0)$ and $|\tilde{n}_{12}|=C_1^{-1}+C_2^{-1}$, $\tilde{\kappa}_{12}=C_1^{-1}C_2$, $\tilde{\omega}(\tilde{\theta})\cdot\tilde{p}=-C_1^{-1}C_2$ when $(\tilde{\theta}_1,\tilde{\theta}_2)=(0,0)$, we can obtain that

$$\begin{aligned}&\tilde{F}(0,0)\\ &=C_1^{-1}C_2\big\{(\lambda+\mu)+\mu C_1C_2^{-1}(1+C_1^{-1}C_2)\\ &\quad+(\lambda+2\mu)(3C_1^{-1}C_2-1)\big\}\\ &=C_1^{-1}C_2\big\{\mu C_1C_2^{-1}+3(\lambda+2\mu)C_1^{-1}C_2\big\}>0.\end{aligned}$$

Therefore we can prove that $\tilde{F}(\tilde{\theta}_1, \tilde{\theta}_2) \neq 0$ provided $|\tilde{\theta} + \tilde{\omega}|$ is different from zero and sufficiently small. Thus the proof is completed. □

Next we observe the non mode-conversion case $(i = 2)$.

Lemma 2.3. *Assume that $|\tilde{\theta} + \tilde{\omega}|$ is sufficiently small. Then we have $m_{11}(\tilde{\theta}, \tilde{\omega}) \neq 0$.*

Proof. Let $i = 2$. According to (2.5), $m_{11}(\tilde{\theta}, \tilde{\omega})$ is expressed as follows:

$$\begin{aligned}
&m_{11}(\tilde{\theta}, \tilde{\omega}) \\
&\quad = \Big[(\lambda + \mu)\tilde{\omega}_1 \big\{2(\tilde{\omega}_1 \tilde{p}_2^{(13)} + \tilde{\kappa}_{12} \tilde{p}_2^{(11)})\tilde{\omega}_1 \tilde{\omega}_3^2 \\
&\qquad + \tilde{p}_2^{(13)}(\tilde{p} \cdot \tilde{q} - \tilde{\omega}_1^2 z_+) + \tilde{p}_2^{(11)}(-\tilde{\omega}_3 \tilde{\omega}_1)(\tilde{\omega} \cdot \tilde{p})\big\} \\
&\qquad + (\lambda + \mu)\tilde{\omega}_2 \big\{2(\tilde{\omega}_2 \tilde{p}_2^{(13)} + \tilde{\kappa}_{12} \tilde{p}_2^{(12)})\tilde{\omega}_1 \tilde{\omega}_3^2 \\
&\qquad + \tilde{p}_2^{(13)}(-\tilde{\omega}_2 \tilde{\omega}_1) z_+ + \tilde{p}_2^{(12)}(-\tilde{\omega}_3 \tilde{\omega}_1)(\tilde{\omega} \cdot \tilde{p})\big\} \\
&\qquad + 2(\tilde{\omega}_3 + \tilde{\kappa}_{12} + C_2|\tilde{n}_{22}|)\big\{\mu(\tilde{\omega}_1 \tilde{p}_2{}^{(11)} + \tilde{\omega}_2 \tilde{p}_2^{(12)}) \\
&\qquad + (\lambda + 2\mu)\tilde{\kappa}_{12} \tilde{p}_2^{(13)}\big\}\tilde{\omega}_1 \tilde{\omega}_3^2 \\
&\qquad + C_2|\tilde{n}_{22}|\big\{\mu \tilde{p}_2^{(11)}(\tilde{p} \cdot \tilde{q} - \tilde{\omega}_1^2 z_+) \\
&\qquad + \mu \tilde{p}_2^{12}(-\tilde{\omega}_2 \tilde{\omega}_1) z_+ \\
&\qquad + (\lambda + 2\mu)\tilde{p}_2^{(13)}(-\tilde{\omega}_3 \tilde{\omega}_1)(\tilde{\omega} \cdot \tilde{p})\big\}\Big] / (\tilde{p} \cdot \tilde{q}).
\end{aligned}$$

Since, in the back-scattering case, we have $\tilde{\omega} = (0, 0, -1)$ and $\tilde{\theta} = (0, 0, 1)$, we can derive that $m_{11}(\tilde{\theta}, \tilde{\omega}) = 2\mu + O(|\tilde{\theta} + \tilde{\omega}|)$.

Therefore, by using our assumption that $|\tilde{\theta} + \tilde{\omega}|$ is sufficiently small, we can prove that $m_{11}(\tilde{\theta}, \tilde{\omega}) \neq 0$. □

As is shown above, since we can prove that $m_{33}(\tilde{\theta}, \tilde{\omega}) \neq 0$ when $i = 1$, and $m_{11}(\tilde{\theta}, \tilde{\omega}) \neq 0$ when $i = 2$, in both cases, it was shown that the leading term of the right hand side of (3.4) dose not vanish.

Thus the proof is completed. □

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