

Dual mean value problem for complex polynomials

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Abstract: We consider an extremal problem for polynomials, which is dual to the well-known Smale mean value problem. We give a rough estimate depending only on the degree.

Key words: Smale's mean value conjecture; critical point.

1. Introduction. Let P be a complex polynomial of degree $d \geq 2$, that is, P is a polynomial map on the complex plane \mathbf{C} of the form

$$P(z) = c_d z^d + c_{d-1} z^{d-1} + \cdots + c_1 z + c_0$$

for complex coefficients c_0, \dots, c_d with $c_d \neq 0$. We denote by \mathcal{P}_d the set of complex polynomials of degree d . We say that $\zeta \in \mathbf{C}$ is a *critical point* of P if $P'(\zeta) = 0$. The image $P(\zeta)$ of a critical point ζ under P is called a *critical value*. We denote by $\text{Crit}(P)$ the set of critical points of P . In 1981, Smale [10] proved the following inequality in connection with root-finding algorithms.

Theorem A (Smale). *Let P be a polynomial of degree $d \geq 2$ over \mathbf{C} and suppose that $z \in \mathbf{C}$ is not a critical point of P . Then there exists a critical point ζ of P such that*

$$(1.1) \quad \left| \frac{P(\zeta) - P(z)}{\zeta - z} \right| \leq 4|P'(z)|.$$

In the same paper, Smale asked whether the factor 4 can be replaced by 1 or even by $1 - 1/d$. See also [9] and [7, §7.2] for background and further references.

To simplify expressions and to emphasize invariance, we introduce some notation. For $\zeta \in \text{Crit}(P)$ and $z \in \mathbf{C} - \text{Crit}(P)$, we define $Q(P, z, \zeta)$ by

$$Q(P, z, \zeta) = \frac{P(z) - P(\zeta)}{(z - \zeta)P'(z)}.$$

For $z \in \text{Crit}(P)$, we extend the definition by setting $Q(P, z, \zeta) = \lim_{z' \rightarrow z} Q(P, z', \zeta)$.

It is easy to verify the invariance relation $Q(\tilde{P}, \tilde{z}, \tilde{\zeta}) = Q(P, z, \zeta)$ for $z = a\tilde{z} + b$, $\zeta = a\tilde{\zeta} + b$ and $\tilde{P}(\tilde{z}) = AP(a\tilde{z} + b) + B$ for constants a, b, A, B with $aA \neq 0$.

We further set

$$S(P, z) = \min\{|Q(P, z, \zeta)| : \zeta \in \text{Crit}(P)\}$$

for $z \in \mathbf{C}$ and

$$K(d) = \sup\{S(P, z) : P \in \mathcal{P}_d, z \in \mathbf{C}\}.$$

Smale's theorem says that $K(d) \leq 4$, while the example $P_0(z) = z^d - dz$ shows that $K(d) \geq 1 - 1/d$. Smale's problem asks to find the value of $K(d)$ and Smale's conjecture can be stated as $K(d) = 1 - 1/d$. This conjecture has been confirmed for degrees $d = 2, 3, 4$ (Tischler [11]), for $d = 5$ (Crane [3]). Sendov and Marinov [8] claim that the conjecture is true for $d \leq 10$ by massive numerical computations, which one cannot check easily. Meanwhile, some improvements were made for Smale's theorem, see [1, 2, 4, 5]. Note that all the known estimates $K(d) \leq \tilde{K}(d)$ satisfy $\liminf_{d \rightarrow \infty} \tilde{K}(d) \geq 4$.

We may pose a dual problem to Smale's mean value problem: Consider the quantity

$$T(P, z) = \max\{|Q(P, z, \zeta)| : \zeta \in \text{Crit}(P)\}$$

for $z \in \mathbf{C}$ and find the value

$$L(d) = \inf\{T(P, z) : P \in \mathcal{P}_d, z \in \mathbf{C}\}.$$

For the polynomial $P^*(z) = (z + 1)^d - 1$, we have $T(P^*, 0) = 1/d$, and hence $L(d) \leq 1/d$.

We should now mention Tischler's strong form of Smale's conjecture:

$$(1.2) \quad \min\left\{\left|Q(P, z, \zeta) - \frac{1}{2}\right| : \zeta \in \text{Crit}(P)\right\} \leq \frac{1}{2} - \frac{1}{d}, \quad P \in \mathcal{P}_d.$$

By the triangle inequality

$$\left| |Q(P, z, \zeta)| - \frac{1}{2} \right| \leq \left| Q(P, z, \zeta) - \frac{1}{2} \right|,$$

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(1.2) would imply the inequality:

$$\max \left\{ S(P, z) - \frac{1}{2}, \frac{1}{2} - T(P, z) \right\} \leq \frac{1}{2} - \frac{1}{d},$$

in other words, $S(P, z) \leq 1 - 1/d$ and $1/d \leq T(P, z)$. Since the inequalities $1 - 1/d \leq K(d)$ and $L(d) \leq 1/d$ trivially hold, the relations $K(d) = 1 - 1/d$ and $L(d) = 1/d$ would follow from (1.2).

It is known that (1.2) is valid for $d \leq 4$, see Tischler [11], where the case $d = 4$ is attributed to J.C. Sikorav. In particular, we have $L(d) = 1/d$ for $d = 2, 3, 4$. Unfortunately, (1.2) does not hold in general. Indeed, Tyson [12] revealed that (1.2) does not hold for every $d \geq 5$.

In the present note, we give the rough estimate $L(d) \geq 1/(d4^d)$ as the first step towards the conjecture $L(d) = 1/d$.

Theorem 1. *Let P be a polynomial of degree $d \geq 2$ over \mathbf{C} and suppose that $z \in \mathbf{C}$ is not a critical point of P . Then there exists a critical point ζ of P such that*

$$\frac{|P'(z)|}{d4^d} \leq \left| \frac{P(\zeta) - P(z)}{\zeta - z} \right|.$$

We proved this theorem when the first author visited Hiroshima University, where the second author was working, in July 2007. We learnt from T.W. Ng that he considered the same problem independently and that he showed the inequality $L(d) > 0$ for each d in the same year by using the notion of amoebae (see [6]). Since there is no explicit lower bound for $L(d)$ in the literature so far, our result seems to be meaningful even though the bound is too far from the conjectured one.

Note also that the conjecture is true for the special case when P is conservative, that is, $P(\zeta) = \zeta$ for every $\zeta \in \text{Crit}(P)$. See the last corollary in [11, p. 455], where Tischler attributes its parent theorem to Yoccoz.

2. Proof of the theorem. By the transformation $\tilde{P}(\tilde{z}) = [P(\tilde{z} + z_0) - P(z_0)]/c_d$, one can easily see that Theorem 1 is equivalent to the following assertion.

Theorem 2. *Let $\zeta_1, \dots, \zeta_{d-1}$ be the critical points of a monic polynomial P of degree $d \geq 2$ with $P'(0) \neq 0$. Then*

$$\max_{1 \leq j \leq d-1} \left| \frac{P(\zeta_j)}{\zeta_j P'(0)} \right| \geq \frac{1}{d4^d}.$$

Proof. Since $P'(z) = d(z - \zeta_1) \cdots (z - \zeta_{d-1})$, the relation

$$(2.1) \quad P'(0) = d(-1)^{d-1} \zeta_1 \cdots \zeta_{d-1}$$

holds.

For $t > 0$, we set $\Delta(t) = \{w \in \hat{\mathbf{C}} : |w| > t\}$, where $\hat{\mathbf{C}}$ stands for the Riemann sphere $\mathbf{C} \cup \{\infty\}$. Let

$$R = \max_{1 \leq j \leq d-1} |P(\zeta_j)|^{1/d}$$

and set $A = \{z \in \mathbf{C} : |P(z)| \leq R^d\}$. Note that A is a continuum containing 0 and all the critical points $\zeta_1, \dots, \zeta_{d-1}$. Since $P : \mathbf{C} - A \rightarrow \Delta(R^d) - \{\infty\}$ is an unbranched covering map of degree d , one can take a single-valued analytic branch $F(w)$ of $P^{-1}(w^d)$ on $\Delta(R)$ with $F(w) = w + O(1)$ as $w \rightarrow \infty$. Note that the function F satisfies the relation

$$(2.2) \quad P(F(w)) = w^d, \quad |w| > R$$

and that $F(\Delta(R)) = \hat{\mathbf{C}} - A$. We now show that F is univalent in $\Delta(R)$. Suppose, to the contrary, that $F(w_0) = F(w_1)$ for distinct points w_0 and w_1 in $\Delta(R)$. Obviously, $z_0 = F(w_0)$ is not the point at infinity. Since P has no critical point in $\mathbf{C} - A$, P is univalent in a small neighborhood V of z_0 . On the other hand, by (2.2), we have $w_0^d = w_1^d$, and thus, $w_1 = \tau w_0$ for a complex number $\tau \neq 1$ with $\tau^d = 1$. We consider the function $F_1(w) = F(\tau w)$ in $|w| > R$. Then $F(w_0) = F_1(w_0)$. Since $F_1(w) = \tau w + O(1)$ as $w \rightarrow \infty$, F_1 and F are not identically equal on $\Delta(R)$. Therefore, $F(w) \neq F_1(w)$ for $w \neq w_0$ close enough to w_0 . By (2.2), we have $P(F(w)) = P(F_1(w))$ for $w \in \Delta(R)$. This is impossible because P is univalent in V . Thus we have shown that F is univalent in $\Delta(R)$.

Let f be a function f on the unit disk $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ by $f(z) = R/F(R/z)$. Since F has no zero in $\Delta(R)$, we see that f is analytic in \mathbf{D} and that $f(0) = f'(0) - 1 = 0$. From what we saw above, we also obtain that f is univalent in \mathbf{D} . The Koebe one-quarter theorem implies that the image $f(\mathbf{D})$ contains the disk $|w| < 1/4$. Since $\zeta_j \in A = \hat{\mathbf{C}} - F(\Delta(R))$, we have

$$(2.3) \quad \frac{R}{|\zeta_j|} \geq \frac{1}{4}, \quad j = 1, 2, \dots, d-1.$$

We may assume that $R = |P(\zeta_1)|^{1/d}$. Then, by (2.1) and (2.3), we have

$$\begin{aligned} & \max_{1 \leq j \leq d-1} \left| \frac{P(\zeta_j)}{\zeta_j P'(0)} \right| \\ & \geq \left| \frac{P(\zeta_1)}{\zeta_1 P'(0)} \right| = \frac{R^d}{|\zeta_1| \cdot d |\zeta_1 \cdots \zeta_{d-1}|} \geq \frac{1}{d4^d}. \end{aligned}$$

The proof is now complete. \square

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