Trace formula and trace identity of twisted Hecke operators on the spaces of cusp forms of weight $k+1/2$ and level $32M$

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Abstract: Let M be an odd positive integer, χ an even quadratic character defined modulo $32M$, and ψ a quadratic primitive character of conductor divisible by 8. Then, we can define twisted Hecke operators $R_\psi\tilde{T}(n^2)$ on the space of cusp forms of weight $k+1/2$, level $32M$, and character χ, under certain conditions on the conductors of χ and ψ. This is a specific feature of the case of half-integral weight. We give explicit trace formulas of the twisted Hecke operators and their trace identities.

Key words: Trace formula; twisting operator; half-integral weight; trace identity; Hecke operator; cusp form.

1. Introduction. Let k and N be positive integers with $4 \mid N$ and χ an even quadratic Dirichlet character defined modulo N. We denote the space of cusp forms of weight $k+1/2$, level N, and character χ by $S(k+1/2, N, \chi)$. Let R_ψ be the twisting operator for a quadratic primitive character ψ and $\tilde{T}(n^2)$ the n^2-th Hecke operator of weight $k+1/2$. In the previous papers [U3] and [U4], we reported trace formulas and trace identities of the twisted Hecke operators $R_\psi\tilde{T}(n^2)$ on $S(k+1/2, N, \chi)$ for various cases. However, we missed one peculiar case of level $32M$ in those papers. Let ψ be a quadratic primitive character whose conductor is divisible by 8. Then $S(k+1/2, 32M, \chi)$ is not generally closed under R_ψ. But, under certain conditions of χ and ψ, R_ψ defines a linear operator on $S(k+1/2, 32M, \chi)$ (See Proposition 1 below). This phenomenon is specific to modular forms of half-integral weight. The aim of this paper is to report explicit trace formulas and trace identities in this case. The details will appear in [U5] or another.

2. Notation. We use the same notation as in the previous paper [U1]. See [U1] and [U2] for the details of notation. Here we explain some of symbols for convenience.

Let k, N, and χ be the same as above. Let a be a non-zero integer and b a positive integer. We write $a \mid b^\infty$ if every prime factor of a divides b.

We denote by $(\frac{\cdot}{\cdot})$ the Kronecker symbol. See [M, p.82] for a definition of this symbol.

Let $\mathbf{H}$ be the complex upper half-plane. Put $j(\gamma, z) = (\frac{-1}{d})^{-1/2}(\frac{c}{d})(cz+d)^{1/2}$ for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$ and $z \in \mathbf{H}$. Let $\mathfrak{G}(k+1/2)$ be the covering group of $GL_2^+(\mathbf{R})$ (cf. [U1, §0(c)]). For a complex-valued function f on $\mathbf{H}$ and $(\alpha, \phi) \in \mathfrak{G}(k+1/2)$, we define a function $f|(\alpha, \phi)$ on $\mathbf{H}$ by: $f|(\alpha,\phi)(z) = \phi(z)^{-1}f(\alpha z)$. By $\Delta_0(N,\chi) = \Delta_0(N,\chi)_{k+1/2}$, we denote the subgroup of $\mathfrak{G}(k+1/2)$ consisting of all pairs $\gamma^* := (\gamma, \phi)$, where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and $\phi(z) = \chi(d)j(\gamma, z)^{k+1/2}$.

Let ρ be any Dirichlet character. We denote the conductor of ρ by $\mathfrak{f}(\rho)$ and for any prime number p, the p-primary component of ρ by ρ_p. Furthermore we set $\rho_A := \prod_{p \mid A} \rho_p$ for an arbitrary positive integer A. Here p runs over all prime divisors of A.

Let V be a finite-dimensional vector space over $\mathbf{C}$. We denote the trace of a linear operator T on V by $\mathrm{tr}(T; V)$.

3. Twisting operator. From now on, we assume that $N = 32M$ with an odd positive integer M and that $\mathfrak{f}(\chi_2)$ divides 4.

Let ψ be a quadratic primitive character of conductor r such that the conductor of ψ_2 is equal to 8. Then we can express the conductor r as $r = 8L$ with a squarefree odd positive integer L.

From now on until the end of this paper, we assume the condition $L^2 \mid M$.

Proposition 1. *Under the above assumptions, the twisting operator R_ψ for ψ*

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$$f=\sum_{n\geqq 1}a(n)q^n\mapsto f\mid R_\psi:=\sum_{n\geqq 1}a(n)\psi(n)q^n$$
$$(q:=\exp(2\pi\sqrt{-1}\,z),\ z\in\mathbf{H})$$

defines a linear operator of $S(k+1/2,32M,\chi)$.

Proof. Take any $f\in S(k+1/2,32M,\chi)$. From $S(k+1/2,32M,\chi)\subset S(k+1/2,64M,\chi)$ and [Sh, Lemma 3.6], we see that $f|R_\psi\in S(k+1/2,64M,\chi)$. Since any element of $S(k+1/2,64M,\chi)$ is fixed by all elements of $\Delta_0(64M,\chi)$, it is sufficient for checking the statement to show that $f|R_\psi$ is fixed by the representative $\begin{pmatrix}1&0\\32M&1\end{pmatrix}^*$ of $\Delta_0(32M,\chi)/\Delta_0(64M,\chi)$.

Now, we put

$$\mathfrak{g}(\psi):=\sum_{i\bmod r}\psi(i)\exp(2\pi\sqrt{-1}\,i/r)$$

and

$$\xi(u):=\left(\begin{pmatrix}r&u\\0&r\end{pmatrix},1\right)\in\mathfrak{G}(k+1/2)$$

for any integer u.

Observing that $\overline{\psi}=\psi$ (because ψ is quadratic), we can express R_ψ as follows (cf. [Sh, Lemma 3.6]):

$$\mathfrak{g}(\psi)\,f\mid R_\psi=\sum_{\substack{u\bmod r\\(u,r)=1}}\psi(u)\,f\mid\xi(u).$$

Hence

$$\text{(1)}\quad \mathfrak{g}(\psi)\,f\mid R_\psi\begin{pmatrix}1&0\\32M&1\end{pmatrix}^*=\sum_{\substack{u\bmod r\\(u,r)=1}}\psi(u)f\mid\xi(u)\begin{pmatrix}1&0\\32M&1\end{pmatrix}^*.$$

For any $u\in(\mathbf{Z}/r\mathbf{Z})^\times$, take $v\in(\mathbf{Z}/r\mathbf{Z})^\times$ such that

$$\text{(2)}\quad \begin{cases}v\equiv u\pmod{L},\\ v\equiv u+4\pmod{8}.\end{cases}$$

By straightforwards caluculation, we have

$$\begin{aligned}\gamma_0&:=\begin{pmatrix}r&u\\0&r\end{pmatrix}\begin{pmatrix}1&0\\32M&1\end{pmatrix}\begin{pmatrix}r&v\\0&r\end{pmatrix}^{-1}\\&=\begin{pmatrix}1+32Mu/r&r^{-2}(r(u-v)-32Muv)\\32M&1-32Mv/r\end{pmatrix}\\&\in\Gamma_0(32M)\end{aligned}$$

and also

$$\begin{aligned}\text{(3)}\quad \xi(u)\begin{pmatrix}1&0\\32M&1\end{pmatrix}^*\xi(v)^{-1}\\=\left(\gamma_0,(32M(z-v/r)+1)^{k+1/2}\right).\end{aligned}$$

Moreover, we can calculate $j(\gamma_0,z)$ as follows: First, since $32Mv/r=4(M/L)v\equiv 0\pmod 4$, we have $(\frac{-1}{1-32Mv/r})=1$. Next, observing that $1-4(M/L)v\equiv 1-4\equiv 5\pmod 8$, we can calculate as follows:

$$\begin{aligned}\left(\frac{32M}{1-32Mv/r}\right)&=\left(\frac{32M}{1-4(M/L)v}\right)\\&=\left(\frac{2M/L^2}{1-4(M/L)v}\right)\\&=\left(\frac{2}{1-4(M/L)v}\right)\left(\frac{M/L^2}{1-4(M/L)v}\right)\\&=\left(\frac{2}{1-4(M/L)v}\right)\left(\frac{M/L^2}{1}\right)\\&=\left(\frac{2}{1-4(M/L)v}\right)=\left(\frac{2}{5}\right)=-1.\end{aligned}$$

Hence, we have

$$\text{(4)}\quad j(\gamma_0,z)=-(32Mz+1-32Mv/r)^{1/2}.$$

On the other hand, since $\mathfrak{f}(\chi_2)\mid 4$ and $\mathfrak{f}(\chi_M)\mid\prod_{p|M}p\mid(M/L)$, we have

$$\begin{aligned}\chi(1-32Mv/r)&=\chi(1-4(M/L)v)\\&=\chi_2(1-4(M/L)v)\,\chi_M(1-4(M/L)v)\\&=\chi_2(1)\,\chi_M(1)=1.\end{aligned}$$

Therefore by (3), (4), and the above, we get

$$\text{(5)}\quad \xi(u)\begin{pmatrix}1&0\\32M&1\end{pmatrix}^*\xi(v)^{-1}=\left(\begin{pmatrix}1&0\\0&1\end{pmatrix},-1\right)\gamma_0^*.$$

Here $\gamma_0^*:=(\gamma_0,\chi(1-32Mv/r)j(\gamma_0,z)^{2k+1})\in\Delta_0(32M,\chi)$. Then, from (1) and (5), we have

$$\begin{aligned}\text{(6)}\quad \mathfrak{g}(\psi)\,f\mid R_\psi\begin{pmatrix}1&0\\32M&1\end{pmatrix}^*&=-\sum_{\substack{u\bmod r\\(u,r)=1}}\psi(u)f\mid\gamma_0^*\,\xi(v)\\&=-\sum_{\substack{u\bmod r\\(u,r)=1}}\psi(u)f\mid\xi(v).\end{aligned}$$

Moreover, we can show from (2)

$$\begin{aligned}\text{(7)}\quad \psi(v)&=\psi_2(v)\psi_L(v)=\psi_2(u+4)\psi_L(u)\\&=-\psi_2(u)\psi_L(u)=-\psi(u).\end{aligned}$$

Since the correspondence $u\mapsto v$ is a permutation of $(\mathbf{Z}/r\mathbf{Z})^\times$, we finally obtain

$$\mathfrak{g}(\psi)\,f\mid R_\psi\begin{pmatrix}1&0\\32M&1\end{pmatrix}^*=\sum_{\substack{v\bmod r\\(v,r)=1}}\psi(v)f\mid\xi(v)$$

$$= \mathfrak{g}(\psi)\, f \mid R_\psi.$$

The proof is completed. □

In the case of $k=1$, we need to make a modification. In this case, the following is well-known (cf. [U1, §0(c)]). The space $S(3/2, 32M, \chi)$ contains a subspace $U(32M;\chi)$ which corresponds to a space of Eisenstein series via the Shimura correspondence. And the subspace $U(32M;\chi)$ is generated by theta series of special type. Let $V(32M;\chi)$ be the orthogonal complement of $U(32M;\chi)$ in $S(3/2, 32M, \chi)$ with respect to the Petersson inner product. Then it is also well-known that $V(32M;\chi)$ corresponds to a space of cusp forms via the Shimura correspondence. Hence we need to consider the subspace $V(32M;\chi)$ in place of $S(3/2, 32M, \chi)$ in the case of $k=1$.

The subspaces $U(32M;\chi)$ and $V(32M;\chi)$ are closed under the twisting operator R_ψ (See [U5] for a proof and refer also to [U2, p.94]). Hence R_ψ gives a linear operator also on the subspace $V(32M;\chi)$. Moreover, the n^2-th Hecke operators $\tilde{T}(n^2)$, $(n, 32M)=1$, also define linear operators on the subspace $V(32M;\chi)$ (cf. [U1, p.508]).

Thus for any positive integer n with $(n, 32M) = 1$, we can consider the twisted Hecke operator $R_\psi\tilde{T}(n^2)$ on the spaces $S(k+1/2, 32M, \chi)$ (if $k \geqq 2$) and $V(32M;\chi)$ (if $k=1$) (cf. [U2, p.86]).

4. Trace formula. Now we state an explicit trace formula of the *twisted Hecke operator* $R_\psi\tilde{T}(n^2)$.

Theorem 1. *Let notation and assumptions be the same as above. Let $\tilde{T}(n^2) = \tilde{T}_{k+1/2,32M,\chi}(n^2)$ be the n^2-th Hecke operator for a positive integer n with $(n, 32M) = 1$* (cf. [U1, §0(c)]). *Then explicit trace formulas of the twisted Hecke operator $R_\psi\tilde{T}(n^2)$ on the spaces $S(k+1/2, 32M, \chi)$ (if $k \geqq 2$) and $V(32M;\chi)$ (if $k=1$) are given as follows*:

$$\mathrm{tr}(R_\psi\tilde{T}(n^2); S(k+1/2, 32M, \chi)) = t(p) + t(e), \quad (\textit{if } k \geqq 2).$$

$$\mathrm{tr}(R_\psi\tilde{T}(n^2); V(32M;\chi)) = t(p) + t(e) + t(d), \quad (\textit{if } k = 1).$$

Here $t(p)$, $t(e)$, $t(d)$ are the contributions from the parabolic, elliptic, and degree part respectively. They are given by the formulas (1.1)–(1.3) *below.*

We use the following notation in those formulas. Let $\mathbf{Z}_+$ be the set of all positive integers. For a real number x, $[x]$ means the greatest integer less than and equal to x. For a prime number p, let $\mathrm{ord}_p(\cdot)$ be the p-adic additive valuation with $\mathrm{ord}_p(p) = 1$ and $|\cdot|_p$ the p-adic absolute value which is normalized with $|p|_p = p^{-1}$. Put $\nu = \nu_p := \mathrm{ord}_p(M)$ for any odd prime number p. And we decompose the level $N = 32M$ with respect to L as follows:

$$N = 32L_0L_2, \quad L_0 > 0, \quad L_2 > 0,$$
$$L_0 \mid L^\infty, \quad (L_2, 2L) = 1.$$

Then we have $L_2 = N\prod_{p|2L}|N|_p$.

(1.1)

$$\begin{aligned} t(p) = (-1)^k\,\psi(-1)^k\, n^{k-1}\,\chi(n) &\\ \times \begin{cases} \chi_2(-1), & \text{if } n \equiv \psi(-1) \pmod 4, \\ 1, & \text{if } n \equiv -\psi(-1) \pmod 4, \end{cases} &\\ \times \prod_{p|L} p^{[(\nu-1)/2]} &\\ \times \prod_{p|L_2}\left(p^{[\nu/2]} + \left(\frac{-8Ln}{p}\right)^{\nu} p^{[(\nu-1)/2]}\right) &\\ \times \sum_{0<a|n_0} h'(-8Ln/a^2). & \end{aligned}$$

Here the notation is as follows: we decompose $n = {n_0}^2 n_1$ (n_0, $n_1 \in \mathbf{Z}_+$, n_1: squarefree). And let $\mathcal{O}(-d)$ be the order of discriminant $-d$ in the imaginary quadratic number field $\mathbf{Q}(\sqrt{-d})$, $h(-d)$ the number of proper ideal classes of the order $\mathcal{O}(-d)$, and $w(-d)$ a half of the number of units in $\mathcal{O}(-d)$. Then put $h'(-d) := h(-d)/w(-d)$.

(1.2)

$$\begin{aligned} t(e) = -\psi(-1)^k\, r^{1-k}\,\chi_L(-n) \times \prod_{p|L} p^{[(\nu-1)/2]} &\\ \times 2\,\chi_2(\psi(-1)) &\\ \times \sum_{\substack{0<s<2\sqrt{rn} \\ s:(*),\, s\equiv 0\,(8)}} \pi_k(s, rn)\, h'(D)\,\alpha_D(m_1') &\\ \times \prod_{p|L_2}(p^{-\mathrm{ord}_p(s)} n_p(\theta_p)). & \end{aligned}$$

Here the condition $(*)$ of s is the following

$$(*) \qquad \mathrm{ord}_p(s) \geqq [(\nu_p + 1)/2] \quad \text{for all prime divisors } p \text{ of } L.$$

The other notation is defined as follows: We decompose $s^2 - 4rn = {m_1}^2 D$ with $m_1 \in \mathbf{Z}_+$ and a discriminant D of an imaginary quadratic field. We put $m_1' := m_1 \prod_{p|N}|m_1|_p$ and $\theta_p := \mathrm{ord}_p(sm_1)$ for a prime number p. Moreover we put a constant $\pi_k(s, rn) := (x^{2k-1} - y^{2k-1})/(x-y)$, where x, y are two roots of the quadratic equation $X^2 - sX + rn = 0$. For a positive integer A, we define a constant $\alpha_D(A)$ by

$$\alpha_D(A) := \prod_{q|A} \left\{ (q^{e+1} - 1) - \left(\frac{D}{q}\right)(q^e - 1) \right\} \bigg/ (q-1),$$

where $A = \prod_{q|A} q^e$ is the prime decomposition of A. The constant $h'(D)$ is the same as in the parabolic part $t(p)$. Finally, the constants $n_p(\theta_p)$ $(p \mid L_2)$ are given by the table below.

Table of $n_p(\theta_p)$.

Case (1) $\quad (p \mid L_2$ and $p \mid s)$

$$\chi_p(r)\chi_p(D) \times n_p(\theta_p) = \begin{cases} p^{\theta_p}\left(p^{[\nu/2]} + \left(\frac{D}{p}\right)^{\nu} p^{[(\nu-1)/2]}\right), & \text{if } \theta_p \geqq [(\nu+1)/2]. \\ \left(1 + \left(\frac{D}{p}\right)\right) p^{2\theta_p}, & \text{if } \theta_p \leqq [(\nu-1)/2]. \end{cases}$$

Case (2) $\quad (p \mid L_2$, $p \nmid s$ and $p \mid D)$

$$\chi_p(r) \times n_p(\theta_p) = \begin{cases} \{(p^{[\nu/2]} + p^{[(\nu-1)/2]})p^{\theta_p+1} - (p^{\nu} + p^{\nu-1})\}(p-1)^{-1}, \\ \qquad \text{if } \theta_p \geqq [\nu/2]. \\ 0, \quad \text{if } \theta_p \leqq [\nu/2] - 1. \end{cases}$$

Case (3) $\quad (p \mid L_2$, $p \nmid s$ and $p \nmid D)$

$$\chi_p(r) \times n_p(\theta_p) = \begin{cases} \left(p - \left(\frac{D}{p}\right)\right)\left(p^{[\nu/2]} + p^{[(\nu-1)/2]}\right)\left(p^{\theta_p} - p^{[\nu/2]}\right) \\ \quad \times (p-1)^{-1} + \left(p^{[\nu/2]} + \left(\frac{D}{p}\right)^{\nu} p^{[(\nu-1)/2]}\right) p^{[\nu/2]}, \\ \qquad \text{if } \theta_p \geqq [(\nu+1)/2]. \\ \left(1 + \left(\frac{D}{p}\right)\right) p^{2\theta_p}, \quad \text{if } \theta_p \leqq [(\nu-1)/2]. \end{cases}$$

(1.3)

$$\begin{aligned} t(d) &= \psi(-1)\,\chi_2(\psi(-1))\,\chi_L(-n)\,\chi_{L_2}(r) \\ &\times \prod_{p|n} \frac{p^{\tau+1} - 1}{p-1} \\ &\times \prod_{p|L_2} \left\{ \left[\frac{\nu_p - \alpha_p}{2}\right] + 1 + \left[\frac{\nu_p + \alpha_p - 1}{2}\right]\left(\frac{-rn}{p}\right) \right\}. \end{aligned}$$

Here the notation is as follows: Let $n = \prod_{p|n} p^{\tau}$ be the prime decomposition of n. For any prime divisor p of L_2, the constant α_p is defined by

$$\chi_p = \left(\frac{\ }{p}\right)^{\alpha_p}, \quad (\alpha_p = 0, 1).$$

This is possible, because χ is a quadratic character and p is odd.

5. Trace identity. Using the above explicit trace formula, we can obtain trace identities between the twisted Hecke operators $R_{\psi}\tilde{T}(n^2)$ and linear combinations of Hecke operators of integral weight and Atkin-Lehner involutions.

We prepare a little more notation for the statement of trace identity.

First we put

$$N_0 := \prod_{p|L} p^{2[(\nu_p - 1)/2]+1}.$$

Here p runs over all prime divisors of L.

Next, let A be any positive integer. For any odd prime number p and any integers a, b $(0 \leqq a \leqq \mathrm{ord}_p(A)/2)$, we put

$$\lambda_p(\chi_p, \mathrm{ord}_p(A); b, a) := \begin{cases} 1, & \text{if } a = 0, \\ 1 + \left(\frac{-b}{p}\right), & \text{if } 1 \leqq a \leqq [(\mathrm{ord}_p(A) - 1)/2], \\ \chi_p(-b), & \text{if } \mathrm{ord}_p(A) \text{ is even} \\ & \text{and } a = \mathrm{ord}_p(A)/2 \geqq 1. \end{cases}$$

Then for any integer b and any *square* integer c, we put

$$\Lambda_{\chi}(\psi, A; b, c) := \prod_{\substack{p|A \\ (p,r)=1}} \lambda_p(\chi_p, \mathrm{ord}_p(A); b, \mathrm{ord}_p(c)/2).$$

Here p runs over all prime divisors of A which are prime to r.

Furthermore, let B be a positive divisor of A such that $B \mid r^{\infty}$ and $(A/B, B) = 1$. For all positive integers n such that $(n, 32M) = 1$, we define

$$\begin{aligned} \mathscr{C}_{\psi}[2k, n; A, B, \chi] &= \mathscr{C}_{\psi}[A, B, \chi] \\ &:= \sum_{\substack{0 < N_1 | A \\ N_1 = \square,\ (N_1, r) = 1}} \Lambda_{\chi}(\psi, A; rn, N_1) \\ &\qquad \times \mathrm{tr}(W(BN_1)T(n); S(2k, N_1N_2)), \end{aligned}$$

where N_1 runs over all square positive divisors of A which are prime to r and $N_2 := A\prod_{p|N_1} |A|_p$. $T(n)$ is the Hecke operator of weight $2k$ and $W(BN_1)$ is the Atkin-Lehner involution. $S(2k, N_1N_2)$ is the space of cusp forms of weight $2k$ and level N_1N_2. See [U1, §0(b) and §3] for the details of these definitions.

Remark. All spaces occuring in the definition of $\mathscr{C}_\psi[A,B,\chi]$ are contained in the space $S(2k,A)$.

Finally, $\chi'_r := \prod_{p|N,(p,r)=1} \chi_p$, where p runs over all prime divisors of $N = 32M$ which are prime to r. Then we put

$$c(k,n;\psi,\chi) = c(\psi,\chi) := \psi(-1)^k \chi_r(n)\chi'_r(-r).$$

Under this notation, we can state trace identities of the twisted Hecke operators $R_\psi \tilde{T}(n^2)$.

Theorem 2. *Let notation and assumptions be the same as above. For all positive integers n such that* $(n,32M)=1$, *we have the following trace identity*:

$$\left\{\begin{array}{ll} \mathrm{tr}(R_\psi \tilde{T}(n^2); S(k+1/2, 32M,\chi)) & \textit{if } k \geqq 2 \\ \mathrm{tr}(R_\psi \tilde{T}(n^2); V(32M;\chi)) & \textit{if } k = 1 \end{array}\right\}$$
$$= c(\psi,\chi) \times \left\{\begin{array}{ll} \chi_2(-1), & \textit{if } n \equiv \psi(-1) \pmod 4 \\ 1, & \textit{if } n \equiv -\psi(-1) \pmod 4 \end{array}\right\}$$
$$\times \mathscr{C}_\psi[8N_0L_2, 8N_0, \chi].$$

6. Concluding remark. We can expect to establish a theory of newforms of half-integral weight by using the above trace indentities. See [U6] for related results for the case of level 2^m.

References

[M] T. Miyake, *Modular forms*, Translated from the Japanese by Yoshitaka Maeda, Springer, Berlin, 1989.

[Sh] G. Shimura, On modular forms of half integral weight, Ann. of Math. (2) **97** (1973), 440–481.

[U1] M. Ueda, The decomposition of the spaces of cusp forms of half-integral weight and trace formula of Hecke operators, J. Math. Kyoto Univ. **28** (1988), no. 3, 505–555.

[U2] M. Ueda, The trace formulae of twisting operators on the spaces of cusp forms of half-integral weight and some trace relations, Japan. J. Math. (N.S.) **17** (1991), no. 1, 83–135.

[U3] M. Ueda, Trace formula of twisting operators of half-integral weight in the case of even conductors, Proc. Japan Acad. Ser. A Math. Sci. **79** (2003), no. 4, 85–88.

[U4] M. Ueda, Trace identities of twisted Hecke operators on the spaces of cusp forms of half-integral weight, Proc. Japan Acad. Ser. A Math. Sci. **80** (2004), no. 7, 131–135.

[U5] M. Ueda, The trace formulae of twisting operators on the spaces of cusp forms of half-integral weight and trace identities II. (in preparation).

[U6] M. Ueda, Newforms of half-integral weight in the case of level 2^m. (Preprint). `http://www.math.nara-wu.ac.jp/personal/ueda/index_e.html`