The abc-theorem, Davenport's inequality and elliptic surfaces

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Abstract: We consider the application of the abc-theorem and Davenport's inequality to elliptic surfaces over the projective line $\mathbf{P}^1$, with special attention to the case of equality in the abc-theorem. Some existence theorem and the finiteness results will be given for certain type of elliptic surfaces.

Key words: abc-theorem; elliptic surface.

Introduction. The main theorem of this paper is the following result for a complex elliptic surface of a given arithmetic genus over the projective line $\mathbf{P}^1$ which is semistable and which has the minimal number of singular fibres.

Theorem 0.1. *For any positive integer χ, (i) there exists a semistable elliptic surface of arithmetic genus χ with a section over $\mathbf{P}^1$ which has the minimal number of singular fibres $N = 2\chi + 2$. (ii) The number of isomorphism classes of such elliptic surfaces is finite. (iii) In each isomorphism class, there exists a surface defined over some algebraic number field. (iv) Any elliptic surface in (i) is "extremal" in the sense that the Mordell-Weil rank $r = 0$ and the Picard number is maximal: $\rho = h^{1,1}$.*

Thus we generalize the well-known results in the case $\chi = 1, 2$ of Beauville [1] ($\chi = 1, N = 4$) and Miranda-Persson [13] ($\chi = 2, N = 6$) to arbitrary arithmetic genus χ. The proof is based on the abc-theorem and Davenport's inequality, and the idea to apply them to such questions as above has been tried ealier in the special case of "maximal singular fibres" [22,23].

The above theorem (=Theorem 3.1) will be proved in §3 after we recall the abc-theorem in §1 and the connection to elliptic surfaces in §2. In §4, we consider a related question to classify elliptic surfaces with a given number of singular fibres.

1. Review of the abc-theorem and Davenport's inequality.

1.1. Statements. Let $k = \mathbf{C}$ be the field of complex numbers. For a polynomial $f = f(t) \in k[t]$, let $N_0(f)$ denote the number of distinct zeroes of $f(t)$; thus we always have $N_0(f) \leq \deg(f)$.

Theorem 1.1 (abc). *Let a, b, c be three polynomials with complex coefficients such that*

$$a + b + c = 0, \quad (a, c) = 1, \quad \deg(abc) > 0. \tag{1.1}$$

Then

$$\max(\deg(a), \deg(b), \deg(c)) \leq N_0(abc) - 1. \tag{1.2}$$

Theorem 1.2 (Davenport). *Let f, g be relatively prime polynomials and let*

$$h := f^3 - g^2. \tag{1.3}$$

Then

$$\deg(f) \leq 2(N_0(h) - 1), \deg(g) \leq 3(N_0(h) - 1). \tag{1.4}$$

Remark. (i) Theorem 1.1 is known as the abc-theorem for $k[t]$ since it is formally analogous to the arithmetic ABC-conjecture (for the integer ring $\mathbf{Z}$, instead of the polynomial ring $k[t]$) which has been formulated in middle 1980's (see e.g. [7,11]). But the geometric version is in fact proven earlier by Stothers [25] (in a more general situation of Riemann surfaces); cf. Mason [12], Silverman [24]. For the sake of completeness, we recall in §1.2 the proof based on the Riemann-Hurwitz relation, which gives crucial information for the case of equality and which is best suited for our application to elliptic surfaces.

(ii) The original Davenport's inequality says that, for any non-constant complex polynomials $f(t)$ and $g(t)$ such that $h = f^3 - g^2 \neq 0$, we have

$$\deg(f) \leq 2(\deg(h) - 1), \deg(g) \leq 3(\deg(h) - 1), \tag{1.5}$$

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without assuming $(f,g)=1$. Davenport's elegant proof in [5] uses only linear algebra (cf. [23]).

1.2. Proofs.

Proof of Theorem 1.1. Given three polynomials a,b,c satisfying (1.1), we may assume (without loss of generality) that

$$n = \deg(a) = \deg(b) \geq \deg(c). \tag{1.6}$$

The idea of the proof is to view the rational function $\phi = -a/c$ as the covering map of Riemann spheres and use the Riemann-Hurwitz relation.

First decompose the complex polynomials a,b,c into distinct linear factors:

(1.7)
$$\begin{cases} a(t) = a_0 \prod_{i=1}^{r_1}(t-\alpha_i)^{e_i}, & \sum_{i=1}^{r_1} e_i = \deg(a) = n \\ b(t) = b_0 \prod_{j=1}^{r_2}(t-\beta_j)^{e'_j}, & \sum_{j=1}^{r_2} e'_j = \deg(b) = n \\ c(t) = c_0 \prod_{k=1}^{r_3}(t-\gamma_k)^{e''_k}, & \sum_{k=1}^{r_3} e''_k = \deg(c) \leq n. \end{cases}$$

By assumption, we have

$$N_0(abc) = r_1 + r_2 + r_3. \tag{1.8}$$

Then the degree n map $w = \phi(t) = -a(t)/c(t)$ from $\mathbf{P}^1_t$ to $\mathbf{P}^1_w$ is ramified as follows: (i) over $w=0$, at the r_1 points $t=\alpha_i$ with (ramification) index e_i; (ii) over $w=1$, at the r_2 points $t=\beta_j$ with index e'_j; (iii) over $w=\infty$, at the r_3 points $t=\gamma_k$ with index e''_k and at $t=\infty$ with index $n-\deg(c)$ in case $\deg(c)<n$; and (iv) possibly over some other points $w \neq 0,1,\infty$.

Applying the Riemann-Hurwitz formula to this situation, we have, in case $\deg(c)<n$,

$$\begin{aligned} 2n-2 &= \sum_i (e_i - 1) + \sum_j (e'_j - 1) \\ &\quad + \sum_k (e''_k - 1) + (n - \deg(c) - 1) + V' \end{aligned} \tag{1.9}$$

where $V' \geq 0$ is the contribution from ramification points of (iv) above. Rewriting this relation, we have

$$\begin{aligned} n &= (r_1 + r_2 + r_3) - 1 - V' \\ &= N_0(abc) - 1 - V'. \end{aligned} \tag{1.10}$$

In case $\deg(c) = n$, we have similarly

$$n = N_0(abc) - 2 - V' < N_0(abc) - 1. \tag{1.11}$$

Thus $n \leq N_0(abc) - 1$ in either case. This proves Theorem 1.1. $\square$

Proof of Theorem 1.2. Applying Theorem 1.1 to $f^3 - g^2 - h = 0$, we have

$$\begin{aligned} \deg(f^3) &\leq N_0(fgh) - 1 \\ &\leq \deg(f) + \deg(g) + N_0(h) - 1. \end{aligned}$$

This implies

$$3\deg(f) \leq \deg(f) + \deg(g) + N_0(h) - 1,$$

and similarly we have

$$2\deg(g) \leq \deg(f) + \deg(g) + N_0(h) - 1.$$

These two inequalities imply the claim (1.4). $\square$

1.3. Case of equality. Let us consider the case of equality in the abc-theorem. Assume that equality $n = N_0(abc) - 1$ holds in the above proof. By (1.10) and (1.11), this is the case if and only if we have $\deg(c) < n$ and $V' = 0$; the latter condition says that the map $\phi : \mathbf{P}^1_t \to \mathbf{P}^1_w$ is unramified outside $\{0,1,\infty\}$.

As is well known, such a map, called a *Belyi map*, has an amazing property that ϕ is a rational function of t with coefficients in $\bar{\mathbf{Q}}$ (up to a change of the coordinate t) (see [2], [10, Ch.2]). Here $\bar{\mathbf{Q}}$ denotes the field of algebraic numbers. Thus the above argument implies:

Theorem 1.3. *Assume that equality holds in Theorem* 1.1:

$$\max(\deg(a), \deg(b), \deg(c)) = N_0(abc) - 1. \tag{1.12}$$

Then, by replacing if necessary $a,b,c \in k[t]$ by some common constant multiples la, lb, lc $(l \neq 0, \in k)$ and by making a coordinate change $t \to \alpha t + \beta$ $(\alpha \neq 0, \beta \in k)$, the coeffcients of a,b,c are algebraic numbers, i.e. we have $a,b,c \in \bar{\mathbf{Q}}[t]$.

The corresponding result for Theorem 1.2 is:

Theorem 1.4. *Suppose that we have, for some integer m,*

$$\deg(f) = 2m, \deg(g) = 3m, N_0(h) = m+1. \tag{1.13}$$

Then, by replacing f,g by $\gamma^2 f$, $\gamma^3 g (\gamma \neq 0, \in k)$ if neccesary, f,g belong to $\bar{\mathbf{Q}}[t]$ up to a coordinate change of t.

A triple $\{f,g,h\}$ such that $f^3 - g^2 = h$ is called a *Davenport-Stothers triple* (*DS-triple* in short) of order m if it satisfies the condition:

$$\deg(f) = 2m, \deg(g) = 3m, \deg(h) = m+1. \tag{1.14}$$

In this case, we have $(f,g) = 1$ and $N_0(h) = h$ by [23, Lemma 3.1]. Hence we have:

Corollary 1.5. *Any DS-triple $\{f,g,h\}$ have algebraic integers as coefficients, provided that t is suitably chosen.*

Example 1.6 (cf. [23, §5])**.** Here is an example of DS-triple of order $m = 1$ with $\mathbf{Q}$-coefficients: $f = t^2 - 1, g = t^3 - 3/2 \cdot t, h = 3/4 \cdot t^2 - 1$. For any $m \leq 4$, there is essentially a unique DS-triple of order m, and $f, g, h \in \mathbf{Q}[t]$ in such cases. For $m = 5$, there are four (essentially distinct) DS-triples of order m, of which two are realized by $f, g, h \in \mathbf{Q}[t]$ but the other two are realized only in $\mathbf{Q}(\sqrt{-3})[t]$.

Example 1.7. Some examples of (1.13), with $N_0(h) \neq h$, are classically known since Klein. Start from the Legendre cubic $x(x-1)(x-t)$ and transform it into the Weierstrass cubic $x^3 - 3f(t)x - 2g(t)$. For this $\{f, g\}$, $h = f^3 - g^2$ is a constant multiple of the discriminant $(t(t-1))^2$. This gives an example of (1.13) for $m = 1$. This "level 2" example can be extended to the case of elliptic modular curves (or surfaces [18]) of level $n = 3, 4, 5$, which satisfy (1.13) with $h = h_0^n$ for some $h_0 \in k[t]$.

2. Elliptic surfaces over $\mathbf{P}^1$.

2.1. Elliptic surfaces $S_{f,g}$ and $\mathcal{S}_h$. Given a triple $\{f, g, h\}$ such that $f^3 - g^2 = h \neq 0$, let $E_{f,g}$ and $\mathcal{E}_h$ be the elliptic curves over $k(t)$, defined by

$$E_{f,g} : y^2 = x^3 - 3f(t)x - 2g(t) \tag{2.1}$$

$$\mathcal{E}_h : Y^2 = X^3 - h(t). \tag{2.2}$$

Then $P = (f, g)$ is an *integral point* of $\mathcal{E}_h$ (in the sense that both coordinates belong to $k[t]$), while the discriminant of $E_{f,g}$ is equal to h up to a constant. We find it useful to consider the pair $E_{f,g}$ and $\mathcal{E}_h$ together (see [23]), which may be called the *Shafarevich partner* of each other, since this interplay of $E_{f,g}$ and $\mathcal{E}_h$ reflects the famous Shafarevich theorem [17] for the finiteness of elliptic curves over $\mathbf{Q}$ with good reduction outside a given finite set of primes.

We denote by $S_{f,g}$ the elliptic surface over $\mathbf{P}^1$ defined by (2.1), i.e. the Kodaira-Néron model of $E_{f,g}/k(t)$. Similarly, we denote by $\mathcal{S}_h$ the elliptic surface defined by (2.2).

Now let S be any elliptic surface over $\mathbf{P}^1$ with a section. Then S is isomorphic to $S_{f,g}$ for some $f, g \in k[t]$ which can be so chosen that if f is divisible by l^4 and g is divisible by l^6 for some $l \in k[t]$, then l must be a constant in k. In the following, we exclude the case where both f, g are in k.

The smallest integer n such that

$$\deg(f) \leq 4n, \quad \deg(g) \leq 6n \tag{2.3}$$

is called the *arithmetic genus* of S, and it is denoted by $\chi = \chi(S)$. This is the main numerical invariant of an elliptic surface over $\mathbf{P}^1$ with a section. For instance, we know from surface theory [9] the following facts:

$$\chi(S) = 1 \Leftrightarrow S \text{ is a rational elliptic surface} \tag{2.4}$$

$$\chi(S) = 2 \Leftrightarrow S \text{ is a K3 elliptic surface.} \tag{2.5}$$

The topological Euler number $e(S)$ of S is equal to:

$$e(S) = 12\chi(S). \tag{2.6}$$

2.2. The number of singular fibres. Let $S = S_{f,g}$ and let $\Phi : S \to \mathbf{P}^1$ be the given elliptic fibration. We freely use the known facts on singular fibres which can be found in [9], [14] or [26]. Denote by $\Sigma \subset \mathbf{P}^1$ the set of points supporting the singular fibres of Φ, and let $N = \#\Sigma$ be the number of singular fibres of S. In the sequel, we always assume that S has non-constant J-invariant and there is a singular fibre at $t = \infty$. Thus we have

$$\infty \in \Sigma \quad \text{and} \quad N - 1 = N_0(h). \tag{2.7}$$

To simplify the statements below, the condition (ss-1) ("semistable minus one") will mean that the elliptic fibration $\Phi : S \to \mathbf{P}^1$ is semistable outside $t = \infty$, i.e. all singular fibres at $t \neq \infty$ are of Kodaira type I_n for some $n = 1, 2, \ldots$.

Theorem 2.1. *Let $S = S_{f,g}$, and assume* (2.7) *and the condition* (*ss*-1)*. Then we have*

$$\deg(f) \leq 2(N-2), \quad \deg(g) \leq 3(N-2). \tag{2.8}$$

Proof. Suppose that $(f, g) = 1$. Then the assertion follows from Theorem 1.2 and (2.7). Hence it is enough to prove the following lemma:

Lemma 2.2. *The condition* (*ss*-1) *is equivalent to* $(f, g) = 1$.

Proof. If we let $d = GCD(f, g)$, then d divides h, i.e. the discriminant of $E_{f,g}$. Assume that $d \neq 1$, and suppose that $t - \alpha$ is a factor of d. Then we have $f(\alpha) = g(\alpha) = 0$, and the equation (2.1) becomes $y^2 = x^3$ at $t = \alpha$. Hence the singular fibre at $t = \alpha$ cannot be semistable. The converse is shown by reversing the argument. □

This proves Theorem 2.1. □

Corollary 2.3. *Under the same assumption as in Theorem* 2.1*, we have*

$$N \geq \begin{cases} 2\chi + 2 & \text{if } N \text{ is even} \\ 2\chi + 1 & \text{if } N \text{ is odd.} \end{cases} \tag{2.9}$$

Proof. This follows immediately from (2.8) and the definition of χ in (2.3). $\square$

Theorem 2.4. *Let N be the number of singular fibres of an elliptic surface $\Phi : S \to \mathbf{P}^1$ with a section. Assume the condition (ss-1). Then (i) we have $N \geq 2\chi + 1$, and if $N = 2\chi + 1$, then the singular fibre at ∞ is not semistable. (ii) If Φ is a semistable elliptic fibration, then we have $N \geq 2\chi + 2$.*

Proof. (i) It follows from the above Corollary that $N \geq 2\chi + 1$. Moreover, if $N = 2\chi + 1$, then

$$\deg(f) \leq 4\chi - 2, \quad \deg(g) \leq 6\chi - 3.$$

In terms of the coodinates at $t = \infty$, the Weierstrass equation (2.1) becomes a cuspidal curve: $y^2 = x^3$ at $t = \infty$. Hence the fibre at ∞ is additive, i.e. it is not semistable. (ii) This is clear from (i). $\square$

Thus we recover Beauville's results [1]:

Corollary 2.5. *For any semistable elliptic surface S over $\mathbf{P}^1$ with a section, the number N of singular fibres is at least 4. Further, if $N = 4$, then S is a rational elliptic surface with 4 fibres of type I_a, I_b, I_c, I_d where $[a, b, c, d]$ is one of the following six cases:* $[1, 1, 1, 9]$, $[1, 1, 2, 8]$, $[1, 1, 5, 5]$, $[1, 2, 3, 6]$, $[2, 2, 4, 4]$, $[3, 3, 3, 3]$.

Proof. The first part is an immediate consequence of Theorem 2.4, since $\chi \geq 1$. If $N = 4$, then we have $\chi = 1$, hence by (2.4), S is a rational elliptic surface. Suppose the 4 fibres are of type I_a, I_b, I_c, I_d. Then its topological Euler number (2.6) is:

$$a + b + c + d = 12. \tag{2.10}$$

On the other hand, the trivial lattice T (cf. [15,19]) is isomorphic to the direct sum of root lattices ([4, Ch.4]) $A_{a-1}, \ldots, A_{d-1}$. Its rank and determinant are

$$\mathrm{rk}T = (a-1) + \cdots + (d-1) = 8, \quad \det T = abcd.$$

Hence T is a sublattice of E_8 of finite index, say ν. Since E_8 is unimodular, we have $\det T = \nu^2$. Thus

$$abcd = \nu^2. \tag{2.11}$$

The integer solutions of (2.10) and (2.11) are easily determined to be the six given above. $\square$

The above case of $N = 4$ is the first example of more general results related to the case of equality in Theorem 1.1 or 1.2 (see §1.3). Let us consider this question in the next section.

Remark. It should be remarked, though we do not need this fact in this paper, that the inequality (2.9) in Corollary 2.3 is a special case of the following more general and stronger result (see [18, Cor. 2.7], [20, §1] and [6, §2.2]).

Theorem 2.6. *Let $\Phi : S \to C$ be an elliptic surface over a curve of any genus g with a section and with nonconstant J-invariant. Let μ (resp. α) be the number of multiplicative (resp. additive) singular fibres of Φ, and set $\tilde{N} = \mu + 2\alpha$. Then we have*

$$\tilde{N} = 2\chi + 2 - 2g + r + \kappa \tag{2.12}$$

where χ is the arithmetic genus of S, r is the Mordell-Weil rank and $\kappa = h^{1,1} - \rho$ is the difference of the Hodge number $h^{1,1}$ and the Picard number ρ of the surface. In particular, since both r and κ are non-negative integers, it implies:

$$\mu + 2\alpha \geq r + 2\chi + 2 - 2g \geq 2\chi + 2 - 2g. \tag{2.13}$$

The last inequality reduces to (2.9) if we let $g = 0$ and rewrite the condition (ss-1) as $N = \mu + \alpha$ with $\alpha \leq 1$.

3. Existence and finiteness theorems.

Theorem 3.1. *For any positive integer χ, (i) there exist semistable elliptic surfaces S with a section over $\mathbf{P}^1$ with $\chi(S) = \chi$ which have the minimal number of singular fibres $N = 2\chi + 2$. (ii) The number of isomorphism classes of such elliptic surfaces S is finite. (iii) In each isomorphism class, there exists some S defined over $\bar{\mathbf{Q}}$. (iv) Any elliptic surface in (i) is "extremal" in the sense that the Mordell-Weil rank $r = 0$ and the Picard number is maximal: $\rho = h^{1,1}$.*

A variant is:

Theorem 3.2. *For any positive integer χ, (i) there exist elliptic surfaces S with $\chi(S) = \chi$ satisfying the condition (ss-1) which have the minimal number of singular fibres $N = 2\chi + 1$. (ii), (iii), (iv): the same assertion as in Theorem 3.1.*

Proof. We prove both theorems together. Let $S = S_{f,g}$. Then any S attaining the minimal number $N = 2\chi + 2$ (or $N = 2\chi + 1$) corresponds to the triple $\{f, g, h\}$ satisfying (1.13) in Theorem 1.4.

As for the existence (i), we have a more precise statement that there exists such an S among $S_{f,g}$ corresponding to the Davenport-Stother triples of order m $\{f, g, h\}$ with $m = 2\chi$ (or $m = 2\chi - 1$) (see §1.3 and [10,23,25]). (ii) This is a consequence of the finiteness of Belyi maps of bounded degree. Namely, for $S = S_{f,g}$, the absolute invariant J is equal to $\phi = -a/c$ in the proof of abc-theorem (§1.2), i.e. $J = f^3/h$ up to constants, and it is a Belyi map of degree $6m = 12\chi$ (or $12\chi - 6$). The number of such maps is obvioully finite, since the mode of ramification at the three points $\{0, 1, \infty\}$ has only a finite number of possibility once the degree is fixed.

(iii) This is immediate from Theorem 1.4.

(iv) By the standard Picard number formula and Lefschetz-Hodge theorem, we have

$$\rho = r + 2 + \mathrm{rk}\ T \le h^{1,1}. \tag{3.1}$$

The trivial lattice T (in case Theorem 3.1) is the direct sum of the root lattices A_{n_i-1} if the singular fibres of S are of type $I_{n_i}(1 \le i \le N)$. Hence we have

$$\begin{aligned}\rho &= r + 2 + \sum_i n_i - N \\ &= r + 2 + 12\chi - (2\chi + 2) = r + 10\chi,\end{aligned}$$

since $\sum_i n_i$ is equal to the Euler number ([9]) which is 12χ by (2.6). On the other hand, the Hodge number $h^{1,1}$ is equal to 10χ, as is easily seen. Therefore we conclude that $r = 0$ and $\rho = h^{1,1}$. (A slight modification should be made in the case of Theorem 3.2, but it is easy.) [N.B. The assertion (iv) is evident if we apply Theorem 2.6, because the assumption implies $r = 0, \kappa = 0$ in (2.14).] $\square$

Example 3.3. The case $\chi = 1$ in Theorem 3.1 is already mentioned in Corollary 2.5. In case $\chi = 2$, S is a K3 surface by (2.5). The semistable elliptic K3 surfaces with $N = 6$ singular fibres are classified by Miranda-Persson [13]. The first member in their list is the K3 surface with "maximal" singular fibre I_{19} (Dynkin type A_{18}), whose defining equation is given in [22]. The defining equations (or the J-invariant as the Belyi map) for all surfaces in the list of [13] are recently given by [3].

For $\chi \ge 3$, Theorem 3.1 and 3.2 are new, as far as we know.

Remark. The number of the isomorphism classes in (iii) of Theorem 3.1 or 3.2 can be enumerated, in principle, by some combinatorial means. In the case of DS-triples of order m, this number is first enumerated by [25] as a function of m by using group theory. In [10], this number is nicely represented as the number of certain graphs on a sphere, drawn in Grothendieck's style of dessins d'enfants. For instance, for $m = 5 (\chi = 3)$, the four DS-triples of order $m = 5$ (cf. Example 1.6 in §1.3) are represented by the four graphs in Fig. 2.28 of [10, p.128]. This approach can be generalized to more general situation, and the question is reduced to certain combinatorial problem.

4. Toward classification of elliptic surfaces with N singular fibres over $\mathbf{P}^1$.

Problem 4.1. *Given a positive integer N, classify all elliptic surfaces with a section over $\mathbf{P}^1$ with N singular fibres.*

The answer is known only for $N \le 4$.

For $N = 1$, there is none. For $N = 2$, there are four types of elliptic surfaces with two singular fibres: $\{II, II^*\}$, $\{III, III^*\}$, $\{IV, IV^*\}$ or $\{I_0^*, I_0^*\}$ in Kodaira's notation [9]. The J-invariant is constant for all cases.

For the first nontrivial case $N = 3$, the classification has been carried out by Schmickler-Hirzebruch [16], and for the case $N = 4$, by Herfurtner [8].

Around the beginning of the millenium, we reconsidered this problem from the viewpoint of "integral points and Mordell-Weil lattices" (cf. [19,21]). Our approach was reported at the conference at Tokyo Univ. (Jan. 2001), with a new purely algebraic proof for $N = 3$ (unpublished). Let us indicate below our method which should work for any N in principle.

(I) For a given N, the most essential case is when J is nonconstant and all singular fibres are reduced except possibly one. This means that, except at one place $t = \infty$, all singular fibres are either semistable (type I_n for some n) or of type II, III, IV. Let us say that the condition

(red-1) ("reduced minus one")

holds if this is the case. In particular, the previous condition (ss-1) implies (red-1). The following result generalizes Theorem 2.1:

Theorem 4.2. *Assume $S = S_{f,g}$ has N singular fibres. Assume the condition* (*red*-1). *Then we have*

$$\deg(f) \le 2(N - 2), \quad \deg(g) \le 3(N - 2). \tag{4.1}$$

Proof. (Outline) Let $d = GCD(f^3, g^2)$ and set

$$F = f^3/d, G = g^2/d, H = h/d. \tag{4.2}$$

Then apply the abc-theorem to $F - G = H$, $(F, G) = 1$, and a little computation gives the result. $\square$

It follows that we obtain the same conclusion as Corollary 2.3 under the weaker assumption (red-1).

This bounds the arithmetic genus $\chi(S)$ of S in terms of N, when S has N singular fibres and satisfies the condition (red-1).

(II) Take any polynomial $h \in k[t]$ such that $N_0(h) = N - 1$. Then the question is to determine f, g satisfying $f^3 - g^2 = h$ and (4.1). This is (almost) equivalent to the question: to find all integral points $P = (X, Y)$ of the elliptic curve $\mathcal{E}_h$ defined by (2.2) (integral in the sense $X, Y \in k[t]$) such that the section (P) is disjoint from the zero-section (O) in $\mathcal{E}_h$. This is the hardest and at the same time most interesting part of the problem. Very precise result can be expected (cf. [19,21,23, §8]).

(III) If any S' with N' singular fibres ($J \neq Const$) is given and if it does not satisfy the condition (red-1), then one can pass from S' to another surface S, by an elementary "twisting" operation (cf. [8]), such that S has $N \leq N'$ singular fibres and satisfies (red-1). This is to replace a pair of non-deduced fibres to a pair of reduced fibres by the rule: $I_n^* \leftrightarrow I_n (n = 0, 1, \ldots), II^* \leftrightarrow IV$, $III^* \leftrightarrow III, IV^* \leftrightarrow II$, and to continue this until at most one non-reduced fibre is left. The choice of S is not unique from S', but it does not matter.

Exercise. For $N = 3$ or $N = 4$, try the above method and compare the results with [16] or [8]. Then try the open case $N = 5$ or 6.

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