Growth functions associated with Artin monoids of finite type

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Abstract: We prove that the growth functions associated with Artin-monoids of finite type*) are rational functions whose numerators is equal to 1. We give an explicit formula for the denominator polynomial $N_M(t)$ and give three conjectures on it: 1. $N_M(t)$ is irreducible up to a factor 1-t, 2. there are l-1 real distinct roots of $N_M(t)$ on the interval (0,1), and 3. the smallest real root on (0,1) is the unique smallest absolute values of all roots of $N_M(t)$.

Key words: Artin group; Artin monoid; growth function; zeros of polynomial.

1. Introduction. Let G_M^+ be the Artin monoid [B-S, §1.2] generated by the letters a_i, $i \in I$ with respect to a Coxeter matrix $M = (m_{ij})_{i,j\in I}$ [B]. That is, G_M^+ is a monoid generated by the letters a_i, $i \in I$ which are subordinate to the relation generated by

$$a_i a_j a_i \cdots = a_j a_i a_j \cdots \quad i, j \in I, \tag{1.1}$$

where both hand sides of (1.1) are words of alternating sequences of letters a_i and a_j of the same length $m_{ij} = m_{ji}$ with the initials a_i and a_j, respectively. More precisely, G_M^+ is the quotient of the free monoid generated by the letters a_i ($i \in I$) by the equivalence relation: two words U and V in the letters are equivalent, if there exists a sequence $U_0 := U, U_1, \cdots, U_m := V$ such that the word U_k ($k = 1, \cdots, m$) is obtained by replacing a phrase in U_{k-1} of the form on LHS of (1.1) by RHS of (1.1) for some $i, j \in I$. We write by $U \doteq V$ if U and V are equivalent. The equivalence class (i.e. an element of G_M^+) of a word W is denoted by the same notation W. By the definition, equivalent words have the same length. Hence, we define the degree homomorphism:

$$\deg : G_M^+ \longrightarrow \mathbf{Z}_{\geq 0} \tag{1.2}$$

by assigning the length to each equivalence class of words.

In [S, §12], we introduced the space of partition functions $\Omega(G_M^+, I)$ for the monoid G_M^+ and, as for the first step to determine it, we asked to determine the space $\Omega(P_{G_M^+,I})$ of opposite series (see §5) of the growth series $P_{G_M^+,I}(t)$ for the Artin monoid of *finite type* M, where the growth series $P_{G_M^+,I}(t)$ is defined by

$$P_{G_M^+,I}(t) := \sum_{n\in\mathbf{Z}_{\geq 0}} \#\{W \in G_M^+ \mid \deg(W) \leq n\}\ t^n. \tag{1.3}$$

In the present paper, we, conjecturally, answer to the question. Namely, in §2, we show that the growth series has an expression

$$P_{G_M^+,I}(t) = \frac{1}{(1-t)N_M(t)}, \tag{1.4}$$

where $N_M(t)$ is a polynomial determined from the Coxeter-Dynkin graph of M(see Lemma 2.1, c.f. [D](1.21) Corollaire). In §3, we show that zeroes of $N_M(t) = 0$ lie in the disc centered at 0 of radius $1 + \varepsilon$ for a small ε. In §4, we conjecture that there are $l - 1$ distinct real roots on the interval $(0, 1)$ and that the smallest one, say r_M, among them is a unique root of the equation $N_W(t) = 0$ taking the smallest absolute values. Finally in §5, we remark that the conjectures imply that $\Omega(P_{G_M^+,I})$ consists of a single element of the form $1/(1 - r_M s)$, where r_M is expressed as the limit of a sequence of rational numbers.

2. Growth series $P_{G_M^+,I}(t)$. For a Coxeter matrix M, consider the *spherical growth series of the monoid* G_M^+:

$$\dot{P}_{G_M^+,I}(t) := \sum_{n\in\mathbf{Z}_{\geq 0}} \#(\deg^{-1}(n))\ t^n, \tag{2.1}$$

so that $P_{G_M^+,I}(t) = \dot{P}_{G_M^+,I}(t)/(1-t)$. The goal of the present section is the following

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*) In the present paper, we call a Coxeter matrix M is of finite type if it is indecomposable and the associated Coxeter group $\overline{G}_M$ [B, Ch.IV §1] is finite.

Theorem 2.1. *Let G^+_M be the Artin monoid with respect to a Coxeter matrix M of finite type. Then the spherical growth series of the monoid is given by the Taylor expansion of the rational function of the form*

$$\dot{P}_{G^+_M,I}(t) = \frac{1}{N_M(t)}, \tag{2.2}$$

where the denominator $N_M(t)$ is a monic polynomial in t given by

$$N_M(t) := \sum_{J\subset I} (-1)^{\#(J)}\, t^{\deg(\Delta_J)}. \tag{2.3}$$

Here, the summation index J runs over all subsets of I, and Δ_J is the fundamental element in G^+_M associated to the set J ([B-S, §5 Definition]. *See also* Lemma-Definition 1 *and* Remark 2.1 *of the present note*).

Remark 2.2. The formula (2.2) can be deduced from a more general formula due to Deligne [D, (1.21)] on a generating function of the number of positive equivalent classes of galleries associated with a simplicial arrangement of real hyperplanes. Since we restrict our attention to Artin monoids and want to be self-contained, we formulate it as in the present form and give a proof of it.

Proof. The proof is achieved by a recursion formula (2.9) on the coefficients of the growth series. For the proof of the formula, we use the method used to solve the word problem for the Artin monoid [B-S, §6.1], which we recall below.

A word U is said to be divisible (from the left) by a word V, and denoted by $V|U$, if there exists a word W such that $U \doteq VW$. There exists an algorithm, which terminates in finite steps, to decide whether $V|U$ or not for given words U and V (see [B-S, §3]). Since $V \doteq V'$, $U \doteq U'$ and $V|U$ implies $V'|U'$, we use the notation "$|$" of divisibility also between elements of the monoid G^+_M.

We have the following basic concepts [B-S, §5 Definition and §6.1]

Lemma-Definition 1. *For any subset $J \subset I$, there exists a unique element $\Delta_J \in G^+_M$, called the* fundamental element*, such that* i) *$a_i|\Delta_J$ for all $i \in J$, and* ii) *if W is a word such that $a_i|W$ for all $i \in J$, then $\Delta_J|W$.*

2. *To a word W, we associate the subset of I:*

$$I(W) := \{i \in I \mid a_i|W\}. \tag{2.4}$$

One has $\Delta_{I(W)}|W$ and if $\Delta_J|W$ then $J \subset I(W)$. If $W \doteq W'$ then $I(W) = I(W')$. Therefore, we use the same notation $I(W)$ for an element W in G^+_M.

We return to the proof of Theorem. For $n \in \mathbf{Z}_{\geq 0}$ and a subset $J \subset I$, put

$$G^+_n := \{W \in G^+_M \mid \deg(W) = n\} \tag{2.5}$$

$$G^+_{n,J} := \{W \in G^+_n \mid I(W) = J\}. \tag{2.6}$$

By the definition, we have the disjoint decomposition:

$$G^+_n = \amalg_{J\subset I}\, G^+_{n,J}, \tag{2.7}$$

where J runs over all subsets of I. Note that $G^+_{n,\emptyset} = \emptyset$ if $n > 0$ but $G^+_{0,\emptyset} = \{\emptyset\} \neq \emptyset$. For any subset J of I, the union $\amalg_{J\subset K\subset I} G^+_{n,K}$, where the index K runs over all subsets of I containing J, is equal to the subset of G^+_n consisting of elements divisible by Δ_J. That is, it is the image of $G^+_{n-\deg(\Delta_J)}$ under the multiplication by Δ_J from the left. On the other hand, since G^+_M is injectively embedded in the Artin group G_M [B-S, Proposition 5.5], the multiplication map of Δ_J is injective. Hence we obtain a bijection: $G^+_{n-\deg(\Delta_J)} \simeq \amalg_{J\subset K\subset I} G^+_{n,K}$. This implies a numerical relation:

$$\#(G^+_{n-\deg(\Delta_J)}) = \sum_{J\subset K\subset I} \#(G^+_{n,K}). \tag{2.8}$$

Then, for $n > 0$, using this formula, we get the recursion relation:

$$\sum_{J\subset I} (-1)^{\#(J)} \#(G^+_{n-\deg(\Delta_J)}) = 0. \tag{2.9}$$

Together with $\#(G^+_0) = 1$ for $n = 0$, this is equivalent to the formula:

$$\dot{P}_{G^+_M,I}(t) N_M(t) = 1. \tag{2.10}$$

This completes the proof of Theorem. □

By the definition (2.3), one has $N_M(1) = \sum_{J\subset I} (-1)^{\#J} = 0$. That is, $N_M(t)$ has the factor $1 - t$. Then, we conjecture the following

Conjecture 1. The polynomial $\tilde{N}_M(t) := N_M(t)/(1-t)$ is irreducible over $\mathbf{Z}$ for any indecomposable Coxeter matrix M of finite type.

The conjecture is positively confirmed (by a use of computer) for the types A_l, B_l, C_l, D_l ($l \leq 30$), E_6, E_7, E_8, F_4, G_2, H_3 and H_4. The following proof for the types $I_2(p)$ ($p \in \mathbf{Z}_{\geq 3}$) is due to S. Yasuda: there is only one root of $\tilde{N}_{I_2(p)}(t) = (1 - 2t + t^p)/(1-t)$ inside the disc $|t| \leq 1$, and any integral factor $f(t)$ of $\tilde{N}_M(t)$ with no root inside $|t| \leq 1$ is ± 1, since the product of all the roots of $f(t)$ is ± 1.

The conjecture is not necessary to determine $\Omega(P_{G^+_M,I})$, but shall play an important role when we study the global space $V(G^+_M, I)$ of limit elements [S, §11.4.10], on which the Galois group of the splitting field of $\tilde{N}_M(t)$ acts.

Remark 2.3. Actually, $\deg(\Delta_J)$ is equal to the number of reflections in the Coxeter group $\overline{G}_{M|_J}$ associated to $M|_J := (m_{ij})_{i,j\in J}$ [B-S, §5.7], which is also equal to the length of the longest element of the Coxeter group $\overline{G}_{M|_J}$.

Remark 2.4. The fact that the growth functions for the Artin monoids are rational functions with the numerator equal to 1 was also observed by P. Xu [X] and M. Fuchiwaki *et al.* [F-F-S-T] for a few examples of type A of low rank.

3. A bound of the zeroes of the polynomial $N_M(t)$. The following lemma gives a bound of the zeroes of the polynomial $N_M(t)$.

Lemma 3.1. *For a Coxeter matrix M, define a numerical invariant*:

$$a_M := \frac{\deg(\Delta_I) - \max\{\deg(\Delta_J) \mid J \subset I,\ J \neq I\}}{\#I}. \tag{3.1}$$

Then, one has

1. *$a_M \geq 1$ for any finite type Coxeter matrix M,*
2. *all the roots of $N_M(t) = 0$ are contained in the open disc of radius $2^{1/a_M}$ centered at the origin.*

Proof. 1. This is shown by using the classification of finite Coxeter groups:

$$\begin{aligned}
&A_{l\geq 1}: \deg(\Delta_{A_l}) = (l+1)l/2 \\
&\qquad \max\{\deg(\Delta_J)\} = l(l-1)/2 \qquad a_{A_l} = 1,\\
&B_{l\geq 2}: \deg(\Delta_{B_l}) = l^2 \\
&\qquad \max\{\deg(\Delta_J)\} = (l-1)^2 \qquad a_{B_l} = (2l-1)/l,\\
&D_{l\geq 4}: \deg(\Delta_{D_l}) = l(l-1) \\
&\qquad \max\{\deg(\Delta_J)\} = (l-1)(l-2) \qquad a_{D_l} = 2(l-1)/l,\\
&E_6: \deg(\Delta_{E_6}) = 36 \\
&\qquad \max\{\deg(\Delta_J)\} = 20 \qquad a_{E_6} = 8/3,\\
&E_7: \deg(\Delta_{E_7}) = 63 \\
&\qquad \max\{\deg(\Delta_J)\} = 36 \qquad a_{E_7} = 27/7,\\
&E_8: \deg(\Delta_{E_8}) = 120 \\
&\qquad \max\{\deg(\Delta_J)\} = 63 \qquad a_{E_8} = 57/8,\\
&F_4: \deg(\Delta_{F_4}) = 24 \\
&\qquad \max\{\deg(\Delta_J)\} = 9 \qquad a_{F_4} = 15/4,\\
&G_2: \deg(\Delta_{G_2}) = 6 \\
&\qquad \max\{\deg(\Delta_J)\} = 1 \qquad a_{G_2} = 5/2,\\
&H_3: \deg(\Delta_{H_3}) = 15 \\
&\qquad \max\{\deg(\Delta_J)\} = 5 \qquad a_{H_3} = 10/3,\\
&H_4: \deg(\Delta_{H_3}) = 60 \\
&\qquad \max\{\deg(\Delta_J)\} = 15 \qquad a_{H_4} = 45/4,\\
&I_2(p_{\geq 3}): \deg(\Delta_{I_2(p)}) = p \\
&\qquad \max\{\deg(\Delta_J)\} = 1 \qquad a_{I_2(p)} = (p-1)/2.
\end{aligned}$$

2. We compare the roots of $N_M(t) = 0$ with those of $t^{\deg(\Delta_I)} = 0$ by Rouché's theorem as follows: Let $t \in \mathbf{C}$ be a point with $|t| = 2^{1/a_M}$. Then

$$\begin{aligned}
&|N_M(t) - (-1)^{\#(I)} t^{\deg(\Delta_I)}| \\
&\quad = \Bigl| \sum_{J\subset I, J\neq I} (-1)^{\#(J)} t^{\deg(\Delta_J)} \Bigr| \\
&\quad \leq \sum_{J\subset I, J\neq I} |t^{\deg(\Delta_J)}| \\
&\quad \leq (2^{\#(I)} - 1)|t|^{\max\{\Delta_J | J\subset I,\ J\neq I\}} < |t|^{\deg(\Delta_I)}.
\end{aligned}$$

Due to Rouché's theorem, the number of roots of $N_M(t) = 0$ in the disc of radius $2^{1/a_M}$ is equal to that of $t^{\deg(\Delta_I)} = 0$. That is, all the roots of $N_M(t) = 0$ are in the disc $\{|t| < 2^{1/a_M}\}$. $\square$

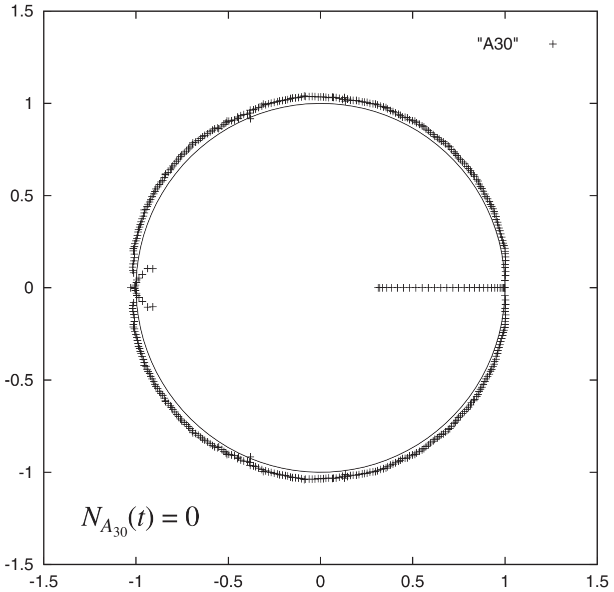
4. Conjectures on the zeroes of the polynomial $N_M(t)$. We give two conjectures on the distribution of the zeroes of $N_M(t)$. We formulate them more than necessary to determine $\Omega(P_{G^+_W,I})$, because of their possible applications to the study of the global space $V(G^+_M, I)$ of limit elements [S, §11].

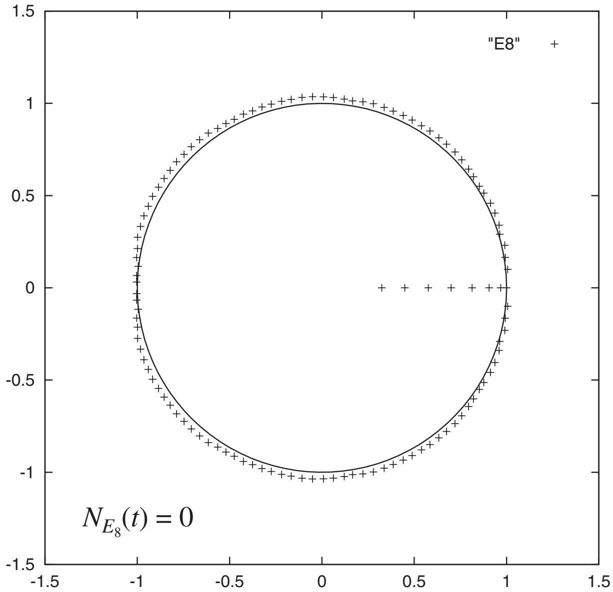
Conjecture 2. There are $(l-1)$ mutually distinct real roots of $N_M(t) = 0$ on the interval $(0, 1)$, where $l := \#I$ is the rank of G_M.

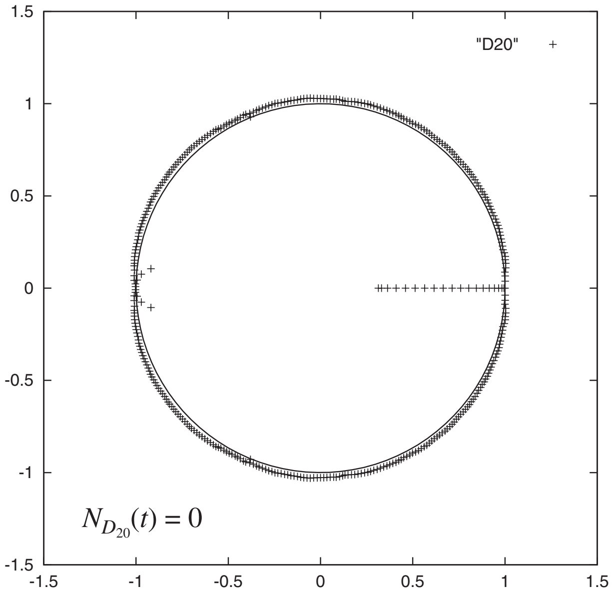
Conjecture 3. Let r_M be the smallest of the real roots in Conjecture 2. Then, the absolute values of the other roots of $N_M(t) = 0$ are larger than r_W.

Conjectures 2 and 3 are positively confirmed (using either Sturm criterion or computer calculations) for the types A_l, B_l, C_l, D_l ($l \leq 30$), E_6, E_7, E_8, F_4, G_2, H_3, H_4 and $I_2(p)$ ($p \geq 3$). Lemma 3.1 together with Conjecture 3 claims that all the roots of $N_M(t) = 0$ lives in the annulus $r_W \leq |t| < 2^{1/a_M}$. Examples show that the angles of the roots are somehow homogeneously distributed (see the following figures). However, we do not know how one can precisely formulate these phenomena. More insights on the distribution of the roots of $N_M(t) = 0$ can be obtained by the following computer drawings of the zeros for the types A_{30}, D_{20} and E_8 due to Shunsuke Tsuchioka (see `http://www.kurims.kyoto-u.ac.jp/~saito/FFST/` for more examples). We observe the followings

Zero loci of $N_M(t)=0$ for the types A_{30}, D_{20} and E_8 (presented by S. Tsuchioka) The zeroes are indicated by $+$.







i) Most of the roots, except for the following 3 cases a), b) and c), are lying outside of the unit circle with rather homogeneous distribution of angles.

a) There are $l-1$ distinct roots on the interval $(0,1)$ and a simple root at $t=1$. Then, roots around $t=1$ are less dense than the other aria.

b) Two horns are glowing at the point $t=-1$ toword the inside of the unit disc. Then, roots around $t=-1$ are less dense than the other aria.

c) For types A and D, there are some warts inside the unit circle (see the Figures).

ii) The values r_M (see Conjecture 3) for the series A_l, B_l and D_l are decreasing as l tends to ∞, but is bounded from below by a positive constant $0.30\ldots$.

Remark 4.1. There is a progress on conjectures 1, 2 and 3 by Seidai Yasuda.

5. The space of opposite series $\boldsymbol{\Omega(P_{G^+_M,I})}$.

We return to the determination of the space $\Omega(P_{G^+_M,I})$ of *opposite series* for $P_{G^+_M,I}$ [S, (11.2.3)], where recall that a series in $\mathbf{R}[[s]]$ is called an opposite series for $P_{G^+_M,I}$ if it is an accumulation point (with respect to the classical topology) of the sequence of polynomials

$$X_n(P_{G^+_M,I}) := \sum_{k=0}^{n} \frac{\#\deg^{-1}(n-k)}{\#\deg^{-1}(n)}\, s^k, \tag{5.1}$$

$$n = 0, 1, 2, \cdots.$$

Let $\Delta^{top}_{P_{G^+_M,I}}(t)$ be the reduced polynomial vanishing at the loci of the poles of $P_{G^+_M,I}$ of the smallest radius and highest order (among them). Actually, Conjectures 2 and 3 imply $\Delta^{top}_{P_{G^+_M,I}}(t) = t - r_M$. Then, Duality Theorem [S, §11 Theorem] says, in general, that if $\deg(\Delta^{top}_{P_{G^+_M,I}}) = h$, then putting $\Delta^{op}_{P_{G^+_M,I}}(s) := s^h \Delta^{top}_{P_{G^+_M,I}}(s^{-1})$, the opposite series have the form $b(s)/\Delta^{op}_{P_{G^+_M,I}}(s)$ for a polynomial $b(s)$ of degree $< h$ (see [S, §11.2 Assertion]). This, in our particular case of $h = 1$, implies that the sequence X_n converges in $\mathbf{R}[[s]]$ to the unique element:

$$a(s) = \frac{1}{1 - r_M s}, \tag{5.2}$$

where the smallest real root $r_M \in \mathbf{R}_{>0}$ is given by

$$r_M = \lim_{n\to\infty} \frac{\#\deg^{-1}(n-1)}{\#\deg^{-1}(n)}. \tag{5.3}$$

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