A characterization of convex cones

By Kyeonghee Jo

Division of Liberal Arts and Sciences, Mokpo National Maritime University,
Mokpo, Chonnam, 530-729, Korea

(Communicated by Masaki Kashiwara, m.j.a., Nov. 12, 2008)

Abstract: Any convex cone has an accumulation point in the base by the action of its automorphism group. In this paper, we prove the converse of this statement, more precisely, a convex domain Ω with a face F of codimension 1 is a cone over F if there is an $\mathrm{Aut}(\Omega)$-orbit accumulating at a point of F.

Key words: Cone; convex domain; projective transformation.

1. Introduction. A convex domain in an affine space is sometimes called a cone even if it is not an affine cone. For example, a triangle and a convex domain $\Omega = \{(x,y,z) \in \mathbf{R}^3 \mid y > x^2, z > 0\}$ can be considered as cones. It's because that a triangle is projectively equivalent to a quadrant and Ω is projectively equivalent to both an elliptic cone $\{(x,y,z) \in \mathbf{R}^3 \mid x^2 + y^2 < z^2, z > 0\}$ and an affine cone $\{(x,y,z) \in \mathbf{R}^3 \mid y/z > (x/z)^2, z > 0\}$. So when a convex domain Ω of an affine space is projectively equivalent to an affine cone, Ω is called a *projective convex cone*, or we say Ω is a *convex cone in projective sense*. In this paper, convex cones will mean projective convex cones instead of affine cones. Note that for each convex cone Ω, we can always choose an affine chart of the projective space where Ω is an affine cone.

Conversely, an affine open cone is projectively equivalent to a convex domain which is the interior of a convex hull of a point and a face of codimension 1 by choosing another affine space. When a convex cone Ω is the interior of the convex hull of a point a and a face F, we say F is a base of a convex cone Ω, or Ω is a *cone over* F (*with a cone point* a).

We can imagine many kinds of convex domains with a face of codimension 1 such as the interior of a half circle and polyhedra. Then what characteristics distinguish convex cones from the others? We can ask if there is any condition about the automorphism group for a domain with a face of codimension 1 to be a cone.

In the mid-twentieth century, the space of quasi-homogeneous domains, domains whose automorphism group have compact fundamental sets, was being researched deeply by several mathematicians such as N. H. Kuiper, J. P. Benzécri, E. B. Vinberg, and J. Vey [1,3,4,6, etc], and several results about characterizations of cones were proved. In 1960, Benzécri [1] proved that *a quasi-homogeneous properly convex domain with a face of codimension* 1 *is a convex cone*, that is, there is no quasi-homogeneous properly convex domain which is not a convex cone but has a face of codimension 1. In 1970, Vey proved that *a quasi-homogeneous properly convex affine domain is an affine cone if it contains an affine open cone* [4]. We can see that Benzécri's result is stronger than Vey's, because a properly convex affine domain containing an affine open cone has an infinite boundary face of codimension 1 when considered as a domain in the projective space.

Every boundary point of a quasi-homogeneous convex domain Ω is an accumulation point of an orbit under the action of its projective automorphism group. So if Ω has a face F of codimension 1, we have a sequence $\{g_i\}$ of projective transformations preserving Ω and a point $x \in \Omega$ such that $g_i(x)$ converges to a point of F. In this paper, we prove that actually the existence of such a sequence of automorphisms is a sufficient condition for a convex domain having a face of codimension 1 to be a cone, no matter whether the domain is quasi-homogeneous:

Theorem 1. *Let* Ω *be a convex domain with a face* F *of codimension* 1*. Then* Ω *is a cone over* F *if the automorphism group of* Ω *has an orbit accumulating at a point of* F*.*

2000 Mathematics Subject Classification. Primary 57S20, 57N16, 52A20, 32F45.

Note that the converse statement of this theorem is obviously true, because every point of the base F is an accumulation point by the action of the automorphism group in case Ω is a cone over F.

2. Basic definitions and lemmas. As mentioned in the introduction, we will consider domains of an affine space in projective category via the well-known equivariant embedding from $(\mathbf{R}^n, \mathrm{Aff}(n, \mathbf{R}))$ into $(\mathbf{RP}^n, \mathrm{PGL}(n+1, \mathbf{R}))$. By this correspondence, a domain Ω of $\mathbf{RP}^n$ will be called *convex* if there exists an affine space $H \subset \mathbf{RP}^n$ such that Ω is a convex subset of H, and $\mathrm{Aut}(\Omega)$, the automorphism group of Ω, will mean the space of all projective transformations preserving Ω. A convex domain Ω is called *properly convex* if it does not contain any complete line.

Definition 2. Let Ω be a properly convex domain of $\mathbf{RP}^n$.

(i) A *face* of Ω is an equivalence class with respect to the equivalence relation given as follows:
  (a) $x \sim y$ if $x \neq y$ and $\overline{\Omega}$ has an open line segment l containing both x and y.
  (b) $x \sim y$ if $x = y$.

(ii) The *support* of a face F, which will be denoted by $\langle F \rangle$, is the projective subspace generated by F.

(iii) Zero dimensional faces are called *extreme points*. Note that p is an extreme point if and only if there is no open line segment which lies in $\partial\Omega$ entirely and contains p.

(iv) Ω is called a *convex sum* of its faces F_1 and F_2, which will be denoted by $\Omega = F_1 \dot{+} F_2$, if $\langle F_1 \rangle \cap \langle F_2 \rangle = \phi$ and Ω is the interior of the convex hull of $\overline{F}_1 \cup \overline{F}_2$ when we consider it as a bounded set in an affine space $\mathbf{A}^n$ in $\mathbf{RP}^n$. When the dimension of F_1 is $n-1$ and F_2 is a point p, Ω is called a cone over F_1 with a cone point p.

Given a properly convex subset F of dimension $n-1$ and a point p which is not contained in the support of F, there are two convex cones over F with a cone point p which are projectively equivalent each other.

Lemma 3. *If a properly convex domain Ω has an $(n-1)$ dimensional face F and an extreme point z which is not contained in $\overline{F}$. Then one of two convex cones over F with a cone point z is contained in Ω and the other does not intersect Ω.*

Proof. Let C_1 and C_2 be two convex cones over F with a cone point z. We can choose a projective hyperplane H in $\mathbf{RP}^n$ such that Ω is a bounded convex set in an affine space $E = \mathbf{RP}^n \setminus H$. Then the interior of the convex hull of F and z in E must be one of two convex cones, say C_1. If we consider E as a vector space with the origin z, we can choose a smallest affine cone C with the origin as a cone point which contains C_1. When we see C_2 in E, it consists of two components, an affine cone $-C$ and $C \setminus \overline{C_1}$. We know C_1 is a subset of Ω. Suppose that there is a point x of Ω inside C_2. Then x is contained in either $-C$ or $C \setminus \overline{C_1}$. If $x \in -C$, then the convex hull of x and F contains z in the interior, which contradicts the fact that $z \in \partial\Omega$. If $x \in C \setminus \overline{C_1}$, then the open line segment with end points x and z intersects F, which contradicts the fact that $F \subset \partial\Omega$. □

Since $\mathrm{PM}(n+1, \mathbf{R})$, which is the projectivization of the group of all (n + 1) by (n + 1) matrices, is a compactification of $\mathrm{PGL}(n+1, \mathbf{R})$, any infinite sequence of non-singular projective transformations contains a convergent subsequence. Note that the limit projective transformation may be singular. For a singular projective transformation g we will denote the projectivization of the kernel and range of g by $Ker(g)$ and $Ran(g)$. Then g maps $\mathbf{RP}^n \setminus Ker(g)$ onto $Ran(g)$.

When g_i is a sequence of projective transformations in $\mathrm{PGL}(n+1, \mathbf{R})$ which converges to a singular projective transformation g, for any compact subset $C \subset \mathbf{RP}^n$ which does not meet $Ker(g)$ the sequence $g_i(C)$ converges uniformly to $g(C)$ [1]. We will need later the following well-known basic facts.

Lemma 4. *Let Ω be a properly convex domain in $\mathbf{RP}^n$ and $\{g_i\} \subset \mathrm{Aut}(\Omega)$. Suppose that the sequence $\{g_i\}$ converges to a singular projective transformation g and that, for a point $x \in \Omega$, the sequence $\{g_i(x)\}$ converges to a point in a face F of Ω. Then*

(i) *$Ker(g) \cap \Omega = \emptyset$ and $Ker(g) \cap \overline{\Omega} \neq \emptyset$,*

(ii) *the interior of $Ker(g) \cap \overline{\Omega}$ in $Ker(g)$ is a face of Ω,*

(iii) *$Ran(g)$ is the support of F,*

(iv) *$g(\Omega) = F$.*

Proof. See [2] for a proof. □

3. The proof of Theorem 1. Any convex domain in $\mathbf{R}^n$ is expressed by the product of $\mathbf{R}^k$ and an $(n-k)$-dimensional properly convex domain. So to prove Theorem 1, it suffices to show the following

Lemma 5. *Let Ω be a properly convex domain in $\mathbf{RP}^n$ and F an $(n-1)$-dimensional face of Ω. Suppose that there is a point x in Ω such that the orbit $\mathrm{Aut}(\Omega)x$ accumulates at a point in F. Then there is a point $\xi \in \partial\Omega$ such that Ω is a convex cone over F with a cone point ξ, i.e.,*

$$\Omega = \{\xi\}\dot{+}F.$$

Proof. By the hypothesis, there is a point p in F and a sequence $\{g_i\}$ in $\mathrm{Aut}(\Omega)$ such that $g_i(x)$ converges to p and the sequence $\{g_i\}$ converges to a projective transformation g as i goes to ∞. Obviously g is singular because $g_i(x)$ converges to a boundary point. By Lemma 4, $Ran(g)$ is the support of F, $g(\Omega) = F$, $Ker(g) \cap \Omega = \emptyset$, and $Ker(g) \cap \overline{\Omega} \neq \emptyset$. Since $Ran(g)$ is $(n-1)$-dimensional, $\dim Ker(g) = 0$. If z is the point in $\partial\Omega$ such that $Ker(g) = \{z\}$, then z must be a zero-dimensional face of Ω. There are two cases.

Case 1: $z \notin \langle F \rangle$.

In this case, $g_i(\overline{F})$ converges uniformly to a subset $g(\overline{F})$ of $\overline{F}$ because $Ker(g) \cap \overline{F} = \emptyset$. Since we can find an open subset U of Ω such that every point x of U is an interior point of the line segment connecting a point of F and z, the interior of $g(\overline{F})$ is not empty in $\langle F \rangle$ and thus $g(\overline{F}) = \overline{F}$. Each of the two convex sums of $\{z\}$ and F is mapped onto F by g and the two cones over $\langle F \rangle \setminus \overline{F}$ with a cone point z (the complement of the closure of the union of two convex sums of $\{z\}$ and F) is mapped onto $\langle F \rangle \setminus \overline{F}$. So Ω should be one of the two convex sums of $\{z\}$ and F because $g(\Omega) = F$ and Ω is connected and convex.

Case 2: $z \in \langle F \rangle$.

In this case, $z \in \partial F$ because $\{z\}$ is a zero-dimensional face of Ω. If b is a boundary point of Ω such that the line segment $\overline{bz}$ contains a point t of Ω, then $g(b) = g(t)$ and $g_i(b)$ converges to $g(b)$. Since $g(t)$ is a point of F, $g_i(b) \in F$ for sufficiently large i. This observation implies that the face containing b must be $(n-1)$-dimensional if b is a boundary point of Ω such that $\overline{bz} \cap \Omega \neq \emptyset$.

Now we consider the set E which consists of such b's, i.e.,

$$\{b \in \partial\Omega \mid \overline{bz} \cap \Omega \neq \emptyset\}.$$

Obviously z is not contained in $\overline{E}$. Here we claim that E is an $(n-1)$-dimensional face of Ω: For each $b \in E$, there is a hyperplane H_b such that $E_b = E \cap H_b$ is an $(n-1)$-dimensional face of Ω which contains b. So E is the disjoint union of all E_b's, i.e., $E = \cup_{b \in E} E_b$. Suppose there are two points b_1 and b_2 in E such that $H_{b_1} \neq H_{b_2}$. Then the open line segment $\overline{b_1 b_2}$ is contained in Ω by convexity. So for each $i = 1, 2$, there is a point a_i in the boundary of E_{b_i} such that $\overline{a_i z} \cap \overline{b_1 b_2} \neq \emptyset$ which means $a_i \in E$. This contradiction completes the proof of our claim.

From Lemma 3, we can choose a convex cone over E with a cone point z inside Ω. So we may denote it by $\{z\}\dot{+}E$. Suppose there exists a point $y \in \Omega \cap \partial(\{z\}\dot{+}E)$. Since $\{z\} \cup \overline{E} \subset \partial\Omega$, we can choose a point $\eta \in \partial E$ such that y is a point of an open line segment $\overline{z\eta}$ and this implies that the boundary point η is an element of E, which is a contradiction. So we can conclude that $\Omega \cap \partial(\{z\}\dot{+}E) = \emptyset$ and thus $\Omega = \{z\}\dot{+}E$. Since $\overline{E} \cap Ker(g) = \emptyset$, $g_i(\overline{E})$ converges uniformly to a subset $g(\overline{E})$ of $\overline{F}$ which has a non-empty interior and this implies that $g_i(E) = F$ for sufficiently large i. From this we see

$$\begin{aligned} \Omega = g_i(\Omega) &= g_i(\{z\}\dot{+}E) \\ &= \{g_i(z)\}\dot{+}g_i(E) = \{g_i(z)\}\dot{+}F \end{aligned}$$

for sufficiently large i, which completes the proof by choosing $g_i(z)$ as ξ. □

If we consider all things in affine category, then we can say more:

Corollary 6. *Let Ω be a convex domain in $\mathbf{R}^n$ and F an $(n-1)$-dimensional face of Ω. Suppose that there is a sequence $\{g_i\}$ of affine transformations which preserve Ω and a point x in the interior of Ω such that the sequence $\{g_i(x)\}$ accumulates at a point of F. Then*

$$\Omega = \mathbf{R}^+ \times F.$$

Proof. Let g be a limit singular projective transformation of the sequence $\{g_i\}$. First, we assume Ω is properly convex. By Lemma 5, there is a point $\xi \in \partial\Omega$ such that $\Omega = \{\xi\}\dot{+}F$ when Ω is considered as a subset of $\mathbf{RP}^n$. So it suffices to show that ξ lies in the infinite boundary of Ω, that is, $\xi \in \partial\Omega \cap \mathbf{RP}^{n-1}_\infty$. As we can see in the proof of Lemma 5, $\{\xi\}$ is either the kernel $Ker(g)$ or the g_i-image of $Ker(g)$ for some i. Since $Ran(g) \cap \mathbf{R}^n$ contains F, $Ker(g)$ is a subset of $\mathbf{RP}^{n-1}_\infty$ by Lemma 3.5 of [2]. This implies that $\xi \in \mathbf{RP}^{n-1}_\infty$ because g_i preserves $\mathbf{R}^n$ for all i.

If Ω contains an affine full line then it is affinely equivalent to $\mathbf{R}^k \times \Omega'$ for some $(n-k)$-dimensional properly convex affine domain Ω' corresponding F to $\mathbf{R}^k \times F'$. Considering the quotient action of the group of affine automorphisms of Ω on Ω', we see that $\Omega' = \mathbf{R}^+ \times F'$ by the argument in the previous paragraph and so we conclude that

$$\Omega = \mathbf{R}^k \times \Omega' = \mathbf{R}^k \times \mathbf{R}^+ \times F' = \mathbf{R}^+ \times F.$$

$\square$

References

[1] J.-P. Benzécri, Sur les variétés localement affines et localement projectives, Bull. Soc. Math. France **88** (1960), 229–332.

[2] K. Jo, Quasi-homogeneous domains and convex affine manifolds, Topology Appl. **134** (2003), no. 2, 123–146.

[3] N. H. Kuiper, On convex locally-projective spaces, in *Convegno Internazionale di Geometria Differenziale, Italia*, 200–213, Ed. Cremonese, Roma, 1953.

[4] J. Vey, Sur les automorphismes affines des ouverts convexes saillants, Ann. Scuola Norm. Sup. Pisa (3) **24** (1970), 641–665.

[5] E. B. Vingberg, The theory of convex homogeneous cones, Trans. Moscow Math. Soc. **12** (1963), 340–403.

[6] È. B. Vinberg and V. G. Kac, Quasi-homogeneous cones, Mat. Zametki **1** (1967), 347–354.